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Test of Emergent Gravity

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Abstract: In this paper we examine a small but detailed test of the emergent gravity picture with explicit solutions in gravity and gauge theory. As a bottom-up approach of emergent gravity recently formulated by us, we derive symplectic U(1) gauge fields starting from the Eguchi-Hanson metric in four-dimensional Euclidean gravity and show that they precisely reproduce U(1) gauge fields of the Nekrasov-Schwarz instanton. As a top-down approach of emergent gravity, we take the U(1) instanton found by Braden and Nekrasov and derive a corresponding gravitational metric. We study the geometrical properties of the four-manifold determined by the Braden-Nekrasov U(1) instanton. It turns out that the gravitational metric of Braden-Nekrasov instanton exhibits a spacetime singularity although it becomes a regular solution without any physical singularity after a Kähler blow up from the gauge theory point of view. This result supports a wishful consensus that a spacetime singularity in general relativity can be resolved in a dual gauge theory description.

Keywords: Models of Quantum Gravity, Gauge-Gravity Correspondence, Non-Commutative Geometry.

Contents

1. Introduction

In order to understand our physical world, it is necessary to take quantum mechanics to be superordinate to classical mechanics. The famous two-slit experiment in quantum mechanics, for example, cannot be explained by simply extrapolating classical physics to the atomic world. Rather classical physics must be understood as phenomena emergent from quantum world when a certain limit is taken to a classical regime. However, in formulating quantum mechanics, we often start by working in a purely classical language that overlays quantum concepts upon the classical framework, that places quantum mechanics in a somewhat secondary position.[1] Fortunately, the strategy of beginning with a theoretical description that is classical and then subsequently including the features of quantum mechanics has been extremely fruitful for many years though it may be too conservative to deal with the measurement problem in quantum mechanics.

But it turns out [2, 3, 4] that the complete formulation of the quantum aspects of spacetime requires a full-fledged quantum theory from the start. In order to get a correct picture on the quantum origin of spacetime, one cannot begin classically and then undergo quantization in the traditional mold. (A similar viewpoint for the complete formulation of string/M-theory was emphasized too in the Chapter 15 of [1].) It is a widely accepted consensus [5] that, in a microscopic scale such as the Planck scale $L_P \sim 10^{-33}$cm where

[1] Here we are open to the conviction [1] (see, especially, Chapter 15) for the standpoint on quantum mechanics.

the quantum effects of spacetime become important, spacetime is no longer commuting but becomes noncommutative (NC), i.e.,

$$[y^\mu, y^\nu] = i\theta^{\mu\nu}. \tag{1.1}$$

According to the above philosophy, one has to regard the Heisenberg algebra (1.1) as a raw precursor to the fabric of spacetime which will be coalesced into an organized form that we recognize as spacetime [2]. Unfortunately, the conventional wisdom is to interpret the NC spacetime (1.1) as an extra structure (e.g., B-fields) defined on a preexisting spacetime. This description inevitably brings about the interpretation that the NC spacetime (1.1) necessarily breaks the Lorentz symmetry. This uneasy picture may be originated from the fact that string theory is not a complete background independent formulation since our present formulation of string theory presupposes the existence of space and time within which strings move about and vibrate. (See the Chapter 15 of [1] for the vivid prospect of background independent formulation of string/M-theory.)

One of the reasons why one should not interpret the NC spacetime (1.1) as an extra structure defined on a preexisting spacetime is ironically coming from the string theory itself. To illuminate this aspect, let us consider a general open string action defined by

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma |dX|^2 - \int_\Sigma B - \int_{\partial\Sigma} A, \tag{1.2}$$

where $X : \Sigma \to M$ is a map from an open string worldsheet Σ to a target spacetime M and $B(\Sigma) = X^*B(M)$ and $A(\partial\Sigma) = X^*A(M)$ are pull-backs of spacetime fields to the worldsheet Σ and the worldsheet boundary $\partial\Sigma$, respectively. The string action (1.2) respects two local gauge symmetries:

(I) Diff(M)-symmetry

$$X \to X' = X'(X) \ \in \ \mathrm{Diff}(M), \tag{1.3}$$

(II) Λ-symmetry

$$(B,\ A) \to (B - d\Lambda,\ A + \Lambda) \tag{1.4}$$

where the gauge parameter Λ is a one-form in M. A simple application of Stokes' theorem to the action (1.2) immediately verifies the symmetry (1.4). Note that the Λ-symmetry is present only when $B \neq 0$. When $B = 0$, the symmetry (1.4) is reduced to $A \to A + d\lambda$, which is the ordinary $U(1)$ gauge symmetry.

The above two local symmetries in string theory must also be realized as the symmetries of low energy effective theory. It is well-known [6] that the low energy effective field theory deduced from the open string action (1.2) describes an open string dynamics on a $(p+1)$-dimensional D-brane worldvolume. For a Dp-brane in closed string background fields, the action describing the resulting low energy dynamics is given by

$$S = \frac{2\pi}{g_s}\int \frac{d^{p+1}x}{(2\pi l_s)^{p+1}}\sqrt{\det\big(g + 2\pi\alpha'(B+F)\big)} + \mathcal{O}\big(l_s\partial F, \cdots\big), \tag{1.5}$$

where $\alpha' = l_s^2$ and $F = dA$ is the field strength of $U(1)$ gauge fields. The DBI action (1.5) respects the two local symmetries, (1.3) and (1.4), as expected. However, Seiberg and

Witten showed [7] that there are two equivalent descriptions, commutative and noncommutative descriptions, of the low energy effective theory, depending on the regularization scheme or path integral prescription for the open string ending on a D-brane. With a point-splitting regularization [7], the spacetime effective action is expressed in terms of NC gauge fields and has the NC gauge symmetry on the NC spacetime defined by (1.1). In this description (which copies the notation and definition in [8]), the low energy effective action is given by

$$\widehat{S} = \frac{2\pi}{G_s} \int \frac{d^{p+1}y}{(2\pi l_s)^{p+1}} \sqrt{\det\left(G + 2\pi\alpha'(\Phi + \widehat{F})\right)} + \mathcal{O}\left(l_s \widehat{D}\widehat{F}, \cdots\right), \tag{1.6}$$

where the NC field strength is defined by

$$\widehat{F}_{\mu\nu} = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - i[\widehat{A}_\mu, \widehat{A}_\nu]_\star. \tag{1.7}$$

The DBI action (1.6) is invariant under the NC gauge transformation

$$\widehat{\delta}_{\widehat{\lambda}} \widehat{A}_\mu = \widehat{D}_\mu \widehat{\lambda} = \partial_\mu \widehat{\lambda} - i[\widehat{A}_\mu, \widehat{\lambda}]_\star. \tag{1.8}$$

The two-form Φ in (1.6) parametrizes some freedom in the description of commutative and NC gauge theories [7, 9] which is defined by

$$\frac{1}{g + 2\pi\alpha' B} = \frac{\theta}{2\pi\alpha'} + \frac{1}{G + 2\pi\alpha'\Phi}. \tag{1.9}$$

Since these two descriptions arise from the same open string theory and the physics should not depend on the regularization scheme, it was argued in [7] that the two descriptions should be equivalent and thus there must be a spacetime field redefinition between ordinary and NC gauge fields, the so-called Seiberg-Witten (SW) map.[2]

An essential point is that Λ-symmetry (1.4) can be considered as par with diffeomorphisms. This fact can be understood as follows [2, 8]. Suppose that the B-field in (1.2) is a symplectic structure on M, i.e., a nondegenerate, closed 2-form. For that case the symplectic structure B defines a bundle isomorphism $B : TM \to T^*M$ by $X \mapsto \Lambda = -\iota_X B$ where ι_X is an interior product with respect to a vector field $X \in \Gamma(TM)$. Then the Λ-transformation in (1.4) can be represented by $B' = B - d\Lambda = B + \mathcal{L}_X B$ where $\mathcal{L}_X$ is a Lie derivative along the flow of X. This means that the Λ-transformation can be identified with a coordinate transformation generated by the vector field X. (See eq. (23) in [3]

[2] If the two descriptions are equivalent, the NC action (1.6) must also respect two local gauge symmetries which correspond to a NC version of the diffeomorphism (1.3) and the Λ-symmetry (1.4). The diffeomorphism symmetry (1.3) may be more accessible by writing the determinant in the action (1.6) as $\det G \exp[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2\pi\alpha')^n}{n} \mathrm{Tr}\widehat{\mathcal{F}}^n]$ where $\widehat{\mathcal{F}}_\mu{}^\nu \equiv (\Phi + \widehat{F})_{\mu\lambda} G^{\lambda\nu}$. It is obvious that $\mathrm{Tr}\widehat{\mathcal{F}}^n$ transforms as $\mathrm{Tr}\Lambda \star \widehat{\mathcal{F}}^n \star \Lambda^{-1}$ under diffeomorphism $\Lambda \in \mathrm{Diff}(M)$ and so it is invariant under the integral. Thus the square root of the determinant in the action (1.6) has the transformation properties of a scalar density under NC diffeomorphisms [10, 11]. The Λ-symmetry (1.4) can be realized with a one-form $\widehat{\Lambda} = \widehat{\Lambda}_\mu(y) dy^\mu$ given by the transformation: $(\Phi, \widehat{A}) \to (\Phi - \widehat{D}\widehat{\Lambda} - i\widehat{\Lambda} \wedge \widehat{\Lambda}, \widehat{A} + \widehat{\Lambda})$ where $\widehat{D}\widehat{\Lambda} \equiv d\widehat{\Lambda} - i(\widehat{A} \wedge \widehat{\Lambda} + \widehat{\Lambda} \wedge \widehat{A})$ and the star-product is implicitly assumed for all formulas. The NC U(1) gauge transformation (1.8) then corresponds to a special case of the NC Λ-symmetry with $\widehat{\Lambda}_\mu = \widehat{D}_\mu \widehat{\lambda}$ while ignoring nonlinear terms $[\widehat{\Lambda}_\mu, \widehat{\Lambda}_\nu]_\star$.

for an explicit verification.) This fact elucidates why Λ-symmetry (1.4) can be regarded as another independent diffeomorphism symmetry. Therefore the low energy field theory described by either (1.5) or (1.6) respects two kinds of diffeomorphism symmetry. However this level of symmetry can be achieved only when the B-field is present and so the B-field greatly enhances the underlying local gauge symmetry, which is unprecedented in theories of particle physics such as the Standard Model.

One can invert the map $B : TM \to T^*M$ to obtain the inverse map $\theta \equiv B^{-1} : T^*M \to TM$ defined by $\alpha \mapsto X = \theta(\alpha)$ such that $X(\beta) = \theta(\alpha, \beta)$ for $\alpha, \beta \in \Gamma(T^*M)$. The bivector $\theta \equiv \frac{1}{2}\theta^{\mu\nu}\frac{\partial}{\partial y^\mu} \wedge \frac{\partial}{\partial y^\nu} \in \Gamma(\wedge^2 TM)$ is called a Poisson structure of M. The Poisson structure defines an $\mathbb{R}$-bilinear operation $\{-,-\}_\theta$, the so-called Poisson bracket [12, 13], given by

$$(f, g) \mapsto \{f, g\}_\theta = \theta(df, dg) = \theta^{\mu\nu}\frac{\partial f}{\partial y^\mu}\frac{\partial g}{\partial y^\nu} \tag{1.10}$$

for smooth functions f, g. Then the real vector space $C^\infty(M)$, together with the Poisson bracket $\{-,-\}_\theta$, forms an infinite-dimensional Lie algebra, called a Poisson algebra $\mathfrak{P} = (C^\infty(M), \{-,-\}_\theta)$. The Dirac quantization of the Poisson algebra $\mathfrak{P} = (C^\infty(M), \{-,-\}_\theta)$ consists of a complex Hilbert space $\mathcal{H}$ and a quantization map $\mathcal{Q}$ to attach to functions $f \in C^\infty(M)$ on M operators $\widehat{f} \in \mathcal{A}_\theta$ acting on $\mathcal{H}$ [14, 15]. The map $\mathcal{Q} : C^\infty(M) \to \mathcal{A}_\theta$ given by $f \mapsto \mathcal{Q}(f) \equiv \widehat{f}$ should be $\mathbb{C}$-linear and an algebra homomorphism:

$$f \cdot g \mapsto \widehat{f \star g} = \widehat{f} \cdot \widehat{g} \tag{1.11}$$

and

$$f \star g \equiv \mathcal{Q}^{-1}\Big(\mathcal{Q}(f) \cdot \mathcal{Q}(g)\Big) \tag{1.12}$$

for $f, g \in C^\infty(M)$ and $\widehat{f}, \widehat{g} \in \mathcal{A}_\theta$. The Poisson structure controls the failure of commutativity

$$[\widehat{f}, \widehat{g}] \sim i\{f, g\}_\theta + \mathcal{O}(\theta^2). \tag{1.13}$$

For example, the coordinate generators $\{y^\mu\}$ of $\mathcal{A}_\theta$ are noncommuting with the Heisenberg algebra relation (1.1). From the deformation quantization point of view, the NC algebra of operators in $\mathcal{A}_\theta$ is equivalent to the deformed algebra of functions defined by the Moyal $\star$-product (1.12) which, according to the Weyl-Moyal map [14, 15], is given by

$$\widehat{f} \cdot \widehat{g} \cong (f \star g)(y) = \exp\Big(\frac{i}{2}\theta^{\mu\nu}\partial_\mu^x \partial_\nu^y\Big) f(x) g(y)|_{x=y}. \tag{1.14}$$

Now we want to carefully contemplate our conventional wisdom imbued with any physical theory all of which describe what happens in a given spacetime. In this mundane picture, the NC spacetime (1.1) is interpreted as an extra structure induced by B-fields defined on a preexisting spacetime and so necessarily breaks the Lorentz symmetry. But, as we emphasized before, the presence of B-fields rather introduces a huge local gauge symmetry (1.4) which is not present in ordinary field theories (without B-fields). Hence we have to ruminate on what had happened in the NC spacetime (1.1). Indeed the enhanced gauge symmetry with $B \neq 0$ gives us a hunch that there will be a radical change of physics– a new physics in NC spacetime. Recently it was shown [2, 3, 4] that the electromagnetism

in NC spacetime can be realized as a theory of gravity and the symplectization of spacetime geometry is the origin of gravity. Remarkably the so-called emergent gravity reveals a novel picture about the origin of spacetime, dubbed as emergent spacetime, which is radically different from the orthodox picture in general relativity. See also recent reviews [16, 17, 18, 19, 20, 21] for some related subjects.

We believe that such a fallacy about NC spacetime hinders the view of the revolutionary aspects of emergent spacetime. In order to appreciate the notion of emergent gravity and correctly contrive quantum gravity based on it, it would be necessary to explicitly show with some examples how the emergent gravity works. In this paper we will examine a tiny yet circumstantial test of the emergent gravity picture with explicit solutions in gravity and gauge theory. As a bottom-up approach of emergent gravity recently formulated by us [22], we will derive symplectic U(1) gauge fields starting from the Eguchi-Hanson metric [23, 24] in four-dimensional Euclidean gravity and show that they precisely reproduce U(1) gauge fields of the Nekrasov-Schwarz instanton [25] derived in [26, 27]. As a top-down approach of emergent gravity, we take the U(1) instanton found by Braden and Nekrasov [28] and derive a corresponding gravitational metric. We will study the geometrical properties of the four-manifold determined by the Braden-Nekrasov U(1) instanton.

The paper is organized as follows. In section 2, we explain how the emergent gravity picture arises from the commutative description of NC gauge theory by the SW map [8]. In particular, we clarify the commutative description of NC U(1) instantons and derive the corresponding self-duality equations for U(1) instantons after the SW map [26, 27, 29]. In section 3, we apply the bottom-up and top-down approaches of emergent gravity to explicit solutions in gravity and gauge theory. First we consider the bottom-up approach of emergent gravity to derive symplectic U(1) gauge fields starting from the Eguchi-Hanson metric [23, 24] in four-dimensional Euclidean gravity and show that they precisely reproduce U(1) gauge fields of the Nekrasov-Schwarz instanton [25]. And then we take the top-down approach with the U(1) instanton found by Braden and Nekrasov [28]. We derive a corresponding gravitational metric and study the geometrical properties of the four-manifold determined by the Braden-Nekrasov U(1) instanton. It turns out that the gravitational metric of Braden-Nekrasov instanton exhibits a spacetime singularity although it becomes a regular solution without any physical singularity after a Kähler blow up from the gauge theory point of view. Our result implies that the commutative description of the Nekarasov-Schwarz instanton is different from the commutative U(1) instanton constructed by Braden and Nekrasov. In section 4, we address the issue on the topological invariants in gravity and U(1) gauge theory. In particular we discuss how to interpret the topological invariants of four-dimensional Riemannian manifolds in terms of U(1) gauge fields in the context of emergent gravity. Finally, in section 5, we summarize the results obtained in this paper and prove the formula (3.19) for generic NC gauge fields. We conclude with a brief discussion about the extension of the bottom-up approach of emergent gravity to general gravitational metrics [22]. In two appendices, we present the definition and several identities for 't Hooft symbols [30] and the explicit forms about the spin connections and curvature tensors for a Riemannian metric we use in this paper.

2. Emergent gravity

From now on, we will focus on a four-dimensional Euclidean space ($p = 3$). It was shown in [31, 32], for slowly varying fields on a single D-brane, that the dual description of the NC DBI action (1.6) through the exact SW map is simply given by the ordinary DBI action (1.5) expressed in terms of open string variables:

$$\begin{aligned}&\int d^4y \sqrt{\det\big(G + \kappa(\Phi + \widehat{F})\big)} \\ &\qquad = \int d^4x \sqrt{\det(1 + F\theta)} \sqrt{\det\big(G + \kappa(\Phi + \mathbf{F})\big)} + \mathcal{O}(l_s \partial F), \end{aligned} \tag{2.1}$$

where $\kappa \equiv 2\pi\alpha' = 2\pi l_s^2$ and

$$\mathbf{F}_{\mu\nu}(x) \equiv \Big(\frac{1}{1 + F\theta} F\Big)_{\mu\nu}(x) \tag{2.2}$$

with the ordinary U(1) field strength defined by

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{2.3}$$

Here slowly varying fields on a D-brane means symplectic gauge fields defined by the commutative limit of NC gauge fields and the field strength of symplectic gauge fields is given by

$$\widehat{F}_{\mu\nu}(y) = \partial_\mu \widehat{A}_\nu(y) - \partial_\nu \widehat{A}_\mu(y) + \{\widehat{A}_\mu, \widehat{A}_\nu\}_\theta(y). \tag{2.4}$$

By comparing both sides of (2.1), one can immediately get the relation between commutative and NC fields given by

$$\widehat{F}_{\mu\nu}(y) = \Big(\frac{1}{1 + F\theta} F\Big)_{\mu\nu}(x), \tag{2.5}$$

$$d^4y = d^4x \sqrt{\det(1 + F\theta)}(x), \tag{2.6}$$

where

$$x^\mu(y) \equiv y^\mu + \theta^{\mu\nu} \widehat{A}_\nu(y). \tag{2.7}$$

An interesting point is that the SW equivalence (2.1) between commutative and NC DBI actions can be derived using only an elementary property, known as the Darboux theorem or the Moser lemma [12, 13], in symplectic geometry. This fact can be explained as follows [8]. Suppose that U(1) gauge theory is defined on a symplectic manifold (M, B) and let us introduce dynamical gauge fields $A_\mu(x)$ fluctuating around the background $B = dA^{(0)}$. The resulting field strength is then given by $\mathcal{F} = B + F$ where $F = dA$ is the curvature two-form of the dynamical gauge field A. One may introduce local coordinates x^a, $a = 1, \cdots, 4$, on a local chart $U \subset M$ where the symplectic structure $\mathcal{F}$ is represented by

$$\mathcal{F} = \frac{1}{2}\Big(B_{ab} + F_{ab}(x)\Big) dx^a \wedge dx^b. \tag{2.8}$$

But one can also introduce another coordinates, say y^μ, on the same local patch $U \subset M$ which are diffeomorphic to x^a, i.e. $x^a = x^a(y)$. Remarkably, the Darboux theorem or

the Moser lemma in symplectic geometry [12, 13] says that it is always possible to find a local coordinate transformation to eliminate the electromagnetic force $F = dA$ in the field strength $\mathcal{F} = B + F$ as long as the space M admits a symplectic structure. In other words, one can find a local coordinate transformation $\phi : x \mapsto y = y(x)$ such that the symplectic structure $\mathcal{F}$ in (2.8) on $U \subset M$ becomes

$$\mathcal{F}|_U = \frac{1}{2} B_{\mu\nu} dy^\mu \wedge dy^\nu. \tag{2.9}$$

If so, it is immediate to see from (2.8) that the so-called Darboux coordinates y^μ will obey the following relation [33, 34]

$$\Big(B_{ab} + F_{ab}(x)\Big) \frac{\partial x^a}{\partial y^\mu} \frac{\partial x^b}{\partial y^\nu} = B_{\mu\nu}. \tag{2.10}$$

By taking the inverse of (2.10), one can rewrite (2.10) in the form

$$\Theta^{ab}(x) \equiv \Big(\frac{1}{B+F}\Big)^{ab}(x) = \theta^{\mu\nu} \frac{\partial x^a}{\partial y^\mu} \frac{\partial x^b}{\partial y^\nu} = \{x^a, x^b\}_\theta(y) \tag{2.11}$$

Using the representation (2.7) for the coordinate transformation $x^a = x^a(y)$, (2.11) reads as

$$\Theta^{ab}(x) = \Big(\theta - \theta \widehat{F} \theta\Big)^{ab}(y) \quad \Leftrightarrow \quad \widehat{F}_{\mu\nu}(y) = \Big(\frac{1}{1+F\theta} F\Big)_{\mu\nu}(x) \tag{2.12}$$

and (2.6) is simply the Jacobian $J = |\frac{\partial y}{\partial x}| = \sqrt{\det(1 + F\theta)}$ of the coordinate transformation $x \mapsto y = y(x)$ which can be derived from (2.10) by taking the determinant on both sides.

Consequently one can see that the SW map in (2.5) and (2.6) can be obtained by the coordinate transformation (2.10) that locally eliminates the electromagnetic force $F = dA$ [32]. In fact, the coordinate transformation (2.10) can also be understood as the Λ-transformation or B-field transformation, $B \to B' = B - d\Lambda$, with $\Lambda = -A$ in (1.4). As we emphasized in section 1, the B-field transformation can be realized as a diffeomorphism $\phi : M \to M$ generated by a vector field X obeying $A = \iota_X B$ and so $F = dA = \mathcal{L}_X B$ and the coordinate transformation (2.10) forms a one-parameter group of diffeomorphisms generated by the flow along X [2]. In the end there exists a novel form of the equivalence principle [2, 8] such that the electromagnetic force can always be eliminated by a local coordinate transformation as long as U(1) gauge theory is defined on a symplectic manifold (M, B). A striking picture then comes out [2, 3, 4] that gravity can emerge from NC U(1) gauge theory as a natural result of the equivalence principle for the electromagnetic force.

Now we will set up a small yet detailed test of the emergent gravity picture using the SW equivalence (2.1). We assume the open string metric $G_{\mu\nu} = \delta_{\mu\nu}$ for simplicity. One can expand both sides of (2.1) into power series of κ. At $\mathcal{O}(\kappa^2)$ one can get the following identity

$$\frac{1}{4} \int d^4 y \mathrm{Tr}(\widehat{F} + \Phi)^2 = \frac{1}{4} \int d^4 x \sqrt{G} \mathrm{Tr}(\mathbf{F} + \Phi)^2, \tag{2.13}$$

where $\mathrm{Tr}(AB) = A_{\mu\nu} B_{\nu\mu}$ and we introduced an effective metric determined by U(1) gauge fields

$$G_{\mu\nu} \equiv \delta_{\mu\nu} + (F\theta)_{\mu\nu}, \qquad G^{\mu\nu} \equiv (G^{-1})^{\mu\nu} = \Big(\frac{1}{1+F\theta}\Big)^{\mu\nu} \tag{2.14}$$

and so (2.2) is given by

$$\mathbf{F}_{\mu\nu} = \left(G^{-1}F\right)_{\mu\nu}. \tag{2.15}$$

Unfortunately it is necessary to introduce many kinds of metrics (e.g., we also need to introduce the gravity metric $g_{\mu\nu}(x)$ in (2.38)) besides the closed and open string metrics already appeared in (1.5) and (1.6). In order not to abuse too many notations for metrics, we have used the same notation for the effective metric (2.14) as the previous constant open string metric in (2.1) which was already assumed to be a flat metric $\delta_{\mu\nu}$. We hope this does not cause too much confusion.

In the usual NC description with $\Phi = 0$, the identity (2.13) takes the form [31, 32]

$$\frac{1}{4}\int d^4y \widehat{F}_{\mu\nu}\widehat{F}^{\mu\nu} = \frac{1}{4}\int d^4x\sqrt{G}G^{\mu\rho}G^{\sigma\nu}F_{\mu\nu}F_{\rho\sigma}, \tag{2.16}$$

while, in the background independent prescription with $\Phi = -B$ [7, 9], the identity (2.13) can be written in the form [29]

$$\frac{1}{4}\int d^4y \{C_\mu, C_\nu\}_\theta^2 = \frac{1}{4}\int d^4x\sqrt{G}G^{\mu\rho}G^{\sigma\nu}B_{\mu\nu}B_{\rho\sigma}, \tag{2.17}$$

where $C_\mu(y) \equiv B_{\mu\nu}x^\nu(y) = B_{\mu\nu}y^\nu + \widehat{A}_\mu(y)$ and we used the relation

$$\{C_\mu, C_\nu\}_\theta = -B_{\mu\nu} + \widehat{F}_{\mu\nu}. \tag{2.18}$$

One can see that the dual description of NC U(1) gauge theory via the exact SW map can be interpreted as the ordinary Maxwell theory coupling to the effective metric (2.14) deformed by U(1) gauge fields [35]. In particular, the background independent description (2.17) clearly shows that the fluctuations of NC photons around the background B-field are mapped through the SW map to the fluctuations of spacetime geometry. It is straightforward to derive the equations of motion [31] for the commutative description in both the cases:

$$\begin{aligned}\Phi = 0: \ \ & \partial_\mu\Big[\sqrt{G}\big\{(\theta G^{-1})^{\mu\nu}\mathrm{Tr}(G^{-1}FG^{-1}F) - 4\big(\theta G^{-1}FG^{-1}FG^{-1}\big)^{[\mu\nu]} \\ & \qquad +4\big(G^{-1}FG^{-1}\big)^{[\mu\nu]}\big\}\Big] = 0, \end{aligned} \tag{2.19}$$

$$\Phi = -B: \ \ \partial_\mu\Big[\sqrt{G}\big\{(\theta G^{-1})^{\mu\nu}\mathrm{Tr}(G^{-1}BG^{-1}B) - 4\big(\theta G^{-1}BG^{-1}BG^{-1}\big)^{[\mu\nu]}\big\}\Big] = 0, \tag{2.20}$$

where $A^{[\mu\nu]} = \frac{1}{2}(A^{\mu\nu} - A^{\nu\mu})$. Indeed the equations of motion for $\Phi = 0$ and $\Phi = -B$ should be equivalent to each other because the difference of two Lagrangians is a constant plus a total derivative term, i.e., $\widehat{S}_{\Phi=-B} - \widehat{S}_{\Phi=0} = \frac{1}{4}\int d^4y B_{\mu\nu}B^{\mu\nu} - \frac{1}{2}\int d^4y \widehat{F}_{\mu\nu}B^{\mu\nu}$ which does not affect the equations of motion. Therefore it may be more convenient to solve the much simpler one (2.20).

Since the field strength of NC U(1) gauge fields is given by (1.7) or (2.4) as its commutative limit and hence is nonlinear due to the commutator term, one can consider a nontrivial solution of the following self-duality equation [25, 36, 37, 38, 39, 40]

$$\widehat{F}_{\mu\nu}(y) = \pm\frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}\widehat{F}_{\rho\sigma}(y). \tag{2.21}$$

A solution of the self-duality equation (2.21) is called a NC U(1) instanton whereas it will be called a symplectic U(1) instanton for the commutative limit where the U(1) field strength is defined by (2.4). But we can apply the commutative description to NC U(1) instantons using the identity (2.16).[3] Using the property $\mathbf{F}_{\mu\nu} = -\mathbf{F}_{\nu\mu}$, it is easy to rewrite the right-hand side of (2.16) in the Bogomolnyi form [26]

$$S_C = \frac{1}{8}\int d^4x\sqrt{G}\Big(\mathbf{F}_{\mu\nu} \mp \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}\mathbf{F}_{\rho\sigma}\Big)^2 \pm \frac{1}{8}\int d^4x\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{2.22}$$

The Bogomolnyi form (2.22) immediately shows that the first term is positive definite while the second term is topological, i.e. a total derivative term and thus does not affect the equations of motion. Hence the minimum of the action $S_{\rm C}$ is achieved in the configurations satisfying the self-duality equation [26]

$$\mathbf{F}_{\mu\nu}(x) = \pm\frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}\mathbf{F}_{\rho\sigma}(x). \tag{2.23}$$

Note that the above equation is directly obtained by applying the exact SW map (2.5) to the NC self-duality equation (2.21). A solution obeying the self-duality equation (2.23) will be called a symplectic U(1) instanton as a commutative limit of NC U(1) instanton.

The metric (2.14) emergent from U(1) gauge fields is in general not symmetric because $G - G^T = F\theta - \theta F \neq 0$. In four dimensions, the six-dimensional vector space $\Lambda^2 T^*M$ of two-forms splits canonically into the sum of three-dimensional vector spaces of self-dual and anti-self-dual two-forms and also the six-dimensional vector space $\Lambda^2 TM$ of bi-vectors splits similarly. So let us take the following decompositions:

$$F_{\mu\nu} = f^{(+)i}\eta^i_{\mu\nu} + f^{(-)i}\overline{\eta}^i_{\mu\nu}, \tag{2.24}$$
$$\theta^{\mu\nu} = \theta^{(+)i}\eta^i_{\mu\nu} + \theta^{(-)i}\overline{\eta}^i_{\mu\nu}, \tag{2.25}$$

where $\eta^i_{\mu\nu}$ and $\overline{\eta}^i_{\mu\nu}$ $(i = 1, 2, 3)$ are self-dual and anti-self-dual 't Hooft symbols, respectively. See the appendix A for the definition and the properties of the 't Hooft symbols. A general condition for the metric (2.14) to be symmetric is given by

$$\varepsilon^{ijk}f^{(+)j}\theta^{(+)k} = 0 = \varepsilon^{ijk}f^{(-)j}\theta^{(-)k}, \quad \forall i = 1, 2, 3. \tag{2.26}$$

This means that $F_{\mu\nu}$ and $\theta^{\mu\nu}$ being second rank tensors of $SO(4) = SU(2)_L \times SU(2)_R$ are parallel to each other in the vector space of $SU(2)_L$ and $SU(2)_R$ Lie algebras. In this case the metric (2.14) becomes symmetric, i.e. $G = G^T$ and so it can be regarded as a usual Riemannian metric.

We will take a self-dual NC space (1.1) that means $\theta^{(-)i} = 0$ in (2.25). It is always possible to rotate $\theta^{(+)i}$ into $(0, 0, \theta^{(+)3})$ such that $\theta^{\mu\nu} = \frac{\theta}{2}\eta^3_{\mu\nu}$. In that case the condition

[3] An instanton in gauge theory is defined as a Euclidean solution with a finite action and so the instanton configuration should approach to a pure gauge at infinity. Our boundary condition is $\widehat{F}_{\mu\nu} \to 0$ at $|y^\mu| \to \infty$ as usual. Thus, when one discuss the instanton solution based on the action (2.17), one has to remove the background part from the action (2.17). It was argued in [29] that there is a background independent as well as gauge covariant way to do this subtraction, which directly shows that self-dual electromagnetism in NC spacetime is equivalent to self-dual Einstein gravity.

(2.26) can be satisfied if $f^{(+)1} = f^{(+)2} = 0$ and the U(1) field strength (2.24) then takes the form

$$F_{\mu\nu} = f^{(+)3}\eta^3_{\mu\nu} + f^{(-)i}\overline{\eta}^i_{\mu\nu}. \tag{2.27}$$

Such U(1) gauge fields result in a usual Riemannian manifold as was shown in [26, 27]. Therefore one can view the right-hand side of (2.16) as U(1) gauge theory defined on a Riemannian manifold whose metric is given by (2.14).[4] Hence one can derive the Bogomolnyi bound for the right-hand side of (2.16) exactly in the same way as a gauge theory defined on a curved manifold:

$$\begin{aligned} S_C = &\frac{1}{8}\int d^4x\sqrt{G}G^{\mu\rho}G^{\nu\sigma}\Big(F_{\mu\nu} \mp \frac{1}{2}\frac{\varepsilon^{\lambda\tau\alpha\beta}}{\sqrt{G}}G_{\mu\lambda}G_{\nu\tau}F_{\alpha\beta}\Big)\Big(F_{\rho\sigma} \mp \frac{1}{2}\frac{\varepsilon^{\xi\eta\gamma\delta}}{\sqrt{G}}G_{\rho\xi}G_{\sigma\eta}F_{\gamma\delta}\Big) \\ &\pm\frac{1}{8}\int d^4x\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \end{aligned} \tag{2.28}$$

Accordingly, the self-duality equation for the action S_C is now given by [8]

$$F_{\mu\nu} = \pm\frac{1}{2}\frac{\varepsilon^{\lambda\tau\rho\sigma}}{\sqrt{G}}G_{\mu\lambda}G_{\nu\tau}F_{\rho\sigma}. \tag{2.29}$$

This equation suggests that symplectic U(1) instantons can be defined as (anti-)self-dual U(1) connections on a four-manifold whose metric is given by (2.14). Actually if we introduce vierbeins of the metric (2.14) such that

$$ds^2 = G_{\mu\nu}(x)dx^\mu \otimes dx^\nu = E^a \otimes E^a, \tag{2.30}$$

the above self-duality equation (2.29) can be written as

$$F = \pm * F \tag{2.31}$$

where $F = \frac{1}{2}F_{ab}E^a \wedge E^b$ and $*$ denotes the Hodge dual operation on forms. Or, in the component form, (2.31) reads as

$$F_{ab} = \pm\frac{1}{2}\varepsilon_{ab}{}^{cd}F_{cd}. \tag{2.32}$$

It is easy to show that (2.32) is equivalent to (2.29) using $F_{ab} = E^\mu_a E^\nu_b F_{\mu\nu}$.

A similar argument can be applied to the background independent description (2.17) although the action (2.17) diverges in general. One can introduce a regularized action by subtracting the most divergent piece (see the footnote 3) and define the theory with the action

$$S_R = \frac{1}{4}\int d^4x\sqrt{G}G^{\mu\rho}G^{\sigma\nu}B_{\mu\nu}B_{\rho\sigma} - \frac{1}{4}\int d^4x B^2_{\mu\nu}. \tag{2.33}$$

Note that the subtraction does not affect the equations of motion and the above regularized action becomes finite. One can then implement the Bogomolyni bound to the regularized

[4]However, it should not be interpreted as a gauge theory defined on a fixed background manifold because the four-dimensional metric (2.14) depends in turn on dynamical U(1) gauge fields.

action (2.33) and the result is simply given by

$$S_R = \frac{1}{8}\int d^4x\sqrt{G}G^{\mu\rho}G^{\nu\sigma}\Big(B_{\mu\nu} \mp \frac{1}{2}\frac{\varepsilon^{\lambda\tau\alpha\beta}}{\sqrt{G}}G_{\mu\lambda}G_{\nu\tau}B_{\alpha\beta}\Big)\Big(B_{\rho\sigma} \mp \frac{1}{2}\frac{\varepsilon^{\xi\eta\gamma\delta}}{\sqrt{G}}G_{\rho\xi}G_{\sigma\eta}B_{\gamma\delta}\Big)$$
$$-\frac{1}{4}\int d^4x\Big(B_{\mu\nu} \mp \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}B_{\rho\sigma}\Big)B^{\mu\nu}. \tag{2.34}$$

This procedure thus leads to another self-duality equation

$$B_{\mu\nu} = \pm\frac{1}{2}\frac{\varepsilon^{\lambda\tau\rho\sigma}}{\sqrt{G}}G_{\mu\lambda}G_{\nu\tau}B_{\rho\sigma} \tag{2.35}$$

which is equivalent, in terms of form language, to

$$B = \pm * B \tag{2.36}$$

with $B = \frac{1}{2}B_{ab}E^a \wedge E^b$. Note that the second term in (2.34) is a total derivative term because $B = dA^{(0)}$ with $A^{(0)}_\mu = -\frac{1}{2}B_{\mu\nu}x^\nu$ and so a boundary term on $\mathbb{S}^3 = \partial\mathbb{R}^4$. On the boundary $\mathbb{S}^3$, the self-duality equation (2.35) reduces to $B_{\mu\nu} = \pm\frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}B_{\rho\sigma}$ and so the second term in S_R identically vanishes.

As was shown before, the SW equivalence (2.1) can be derived by the coordinate transformation (2.10) that locally eliminates the electromagnetic force $F = dA$. This implies that there exists a novel form of the equivalence principle even for the electromagnetic force as long as the U(1) gauge theory is equipped with a symplectic structure B. The quantization of the resulting U(1) gauge theory according to the quantization map (1.11) brings about the NC spacetime (1.1) and results in NC U(1) gauge theory. Consequently, the equivalence principle for the electromagnetic force guarantees that gravity can emerge from NC U(1) gauge theory [2]. Hence a natural question is what kind of four-manifold arises from a solution of the self-duality equation (2.21) known as NC U(1) instantons [25]. In this paper we will focus on its commutative limit satisfying the self-duality equation (2.23) called symplectic U(1) instantons, which can be written in the form (2.29) as long as the metric (2.14) for the solution of (2.23) is symmetric, as we have demonstrated before. It was shown [26, 27, 29] that the equation (2.23) describes gravitational instantons obeying the self-dual equations [41, 42]

$$R_{abef} = \pm\frac{1}{2}\varepsilon_{ab}{}^{cd}R_{cdef}, \tag{2.37}$$

where R_{abcd} is a Riemann curvature tensor. More precisely, if one identifies from the effective metric (2.14) a gravitational metric defined by

$$G_{\mu\nu}(x) = \frac{1}{2}(\delta_{\mu\nu} + g_{\mu\nu}(x)), \tag{2.38}$$

the metric $g_{\mu\nu}(x)$ describes a Ricci-flat Kähler manifold obeying (2.37). In other words, the four-manifold whose metric is given by

$$ds^2 = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu = e^a \otimes e^a \tag{2.39}$$

is a hyper-Kähler manifold [29]. We warn the gravitational metric (2.39) should not be confused with the constant closed string metric in (1.5). In next section we will verify the emergent gravity picture with explicit solutions in gravity and gauge theory.

3. Four-manifolds and U(1) gauge fields

In section 2, we explained why the Riemannian metric (2.39) arises from U(1) gauge fields on a symplectic manifold (M, B) and how to determine it by solving the equations of motion for the U(1) gauge fields. But the emergent gravity can also be inverted, as recently formulated in [22], such that one gets U(1) gauge fields using the relation (2.14) whenever a Riemannian metric (M, g) is given. Now we will illustrate how the emergent gravity works for both the top-down and the bottom-up approaches. For that purpose, we will take an explicit solution in general relativity whose metric is assumed to be of the form

$$ds^2 = A^2(r)(dr^2 + r^2\sigma_3^2) + B^2(r)r^2(\sigma_1^2 + \sigma_2^2) \tag{3.1}$$

and so the covectors (vierbeins) are given by

$$e^1 = B(r)r\sigma^1, \quad e^2 = B(r)r\sigma^2, \quad e^3 = A(r)r\sigma^3, \quad e^4 = A(r)dr. \tag{3.2}$$

We have introduced a left-invariant coframe $\{\sigma^i : i = 1, 2, 3\}$ for $\mathbb{S}^3$ defined by [41]

$$\sigma^i = -\frac{1}{r^2}\eta^i_{\mu\nu}x^\mu dx^\nu \tag{3.3}$$

where $r^2 = x_1^2 + \cdots + x_4^2$. They obey the following structure equations

$$d\sigma^i = -\varepsilon^{ijk}\sigma^j \wedge \sigma^k. \tag{3.4}$$

In appendix B, we present the explicit results for the spin connections and curvature tensors determined by the metric (3.1).

We will assume that the metric (3.1) is asymptotically locally Euclidean (ALE), i.e., $A(r) = B(r) \to 1$ as $r \to \infty$. In that case, it will be useful to introduce the Hopf map $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ which can be represented in terms of $\mathbb{R}^4$ variables as [26]

$$\begin{aligned} T^1 &= -(x^1x^3 + x^2x^4), \\ T^2 &= x^1x^4 - x^2x^3, \\ T^3 &= \frac{1}{2}(x_1^2 + x_2^2 - x_3^2 - x_4^2) \end{aligned} \tag{3.5}$$

and

$$\sum_{i=1}^{3} T^iT^i = \frac{r^4}{4}. \tag{3.6}$$

The following relations may be useful for later purpose (see (3.32) in [26]):

$$\overline{\eta}^i_{\mu\nu}\partial_\nu T^i = 3\eta^3_{\mu\nu}x^\nu, \qquad \overline{\eta}^i_{\mu\nu}x^\nu T^i = \frac{r^2}{2}\eta^3_{\mu\nu}x^\nu. \tag{3.7}$$

3.1 U(1) instanton from Eguchi-Hanson metric

The Eguchi-Hanson metric [23, 24] describes a non-compact, self-dual, ALE space on the cotangent bundle of 2-sphere $T^*\mathbb{S}^2$ with SU(2) holonomy group. The explicit form of the metric is given by

$$ds^2 = f^{-1}(\rho)d\rho^2 + \rho^2\big(\sigma_1^2 + \sigma_2^2 + f(\rho)\sigma_3^2\big) \tag{3.8}$$

where $f(\rho) = 1 - \frac{t^4}{\rho^4}$ and $\rho^4 = r^4 + t^4$. Thus the Eguchi-Hanson metric takes the form (3.1) with

$$A^2(r) = \frac{r^2}{\sqrt{r^4+t^4}} = B^{-2}(r). \tag{3.9}$$

In order to write the metric in terms of the Cartesian coordinates $\{x^\mu\}$,[5] let us plug (3.3) in (3.8). The result can be written as

$$\begin{aligned} ds^2 &= g_{\mu\nu}(x)dx^\mu \otimes dx^\nu \\ &= \Big[\frac{\sqrt{r^4+t^4}}{2r^2}\big(f(r)+1\big)\delta_{\mu\nu} - \frac{\sqrt{r^4+t^4}}{r^4}\big(f(r)-1\big)\big(\eta^3\bar{\eta}^i\big)_{\mu\nu}T^i\Big]dx^\mu \otimes dx^\nu \end{aligned} \tag{3.10}$$

after using the identity

$$x^\mu x^\nu + \eta^3_{\mu\rho}\eta^3_{\nu\sigma}x^\rho x^\sigma = \frac{r^2}{2}\delta^{\mu\nu} - \big(\eta^3\bar{\eta}^i\big)_{\mu\nu}T^i \tag{3.11}$$

which can be checked by a straightforward calculation. Later we will also use the following identity

$$\eta^3_{\mu\rho}x^\rho x^\nu - \eta^3_{\nu\rho}x^\mu x^\rho = \frac{r^2}{2}\eta^3_{\mu\nu} + \bar{\eta}^i_{\mu\nu}T^i \tag{3.12}$$

which can be derived from (3.11) by multiplying $\eta^3_{\mu\rho}$.

Now it is straightforward to identify U(1) gauge fields from the Eguchi-Hanson metric (3.10). Combining (2.14) and (2.38) leads to the relation

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + 2(F\theta)_{\mu\nu}. \tag{3.13}$$

For our choice $\theta^{\mu\nu} = \frac{\theta}{2}\eta^3_{\mu\nu}$ where we put $\theta = 1$ for simplicity, the metric (3.10) leads to the U(1) field strength

$$\begin{aligned} F_{\mu\nu}(x) &= \frac{t^4}{\rho^2 r^4}\bar{\eta}^i_{\mu\nu}T^i - \frac{(\rho^2-r^2)^2}{2\rho^2 r^2}\eta^3_{\mu\nu} \\ &= \frac{t^4}{r^6\sqrt{1+\frac{t^4}{r^4}}}\bar{\eta}^i_{\mu\nu}T^i - \frac{\Big(\sqrt{1+\frac{t^4}{r^4}}-1\Big)^2}{2\sqrt{1+\frac{t^4}{r^4}}}\eta^3_{\mu\nu}. \end{aligned} \tag{3.14}$$

The above result is exactly the same as the field strength of symplectic U(1) gauge fields (see Eq. (3.25) in [26]) determined by solving the self-duality equation (2.23). It was

[5] In order to avoid a confusion, we want to point out that the Cartesian coordinates $\{x^\mu\}$ in the one-form (3.3) and the Hopf map (3.5) should be regarded as the coordinates on the flat space $\mathbb{R}^4$. Therefore it is not necessary to concern about raising and lowering the indices $\mu, \nu, \cdots$ in the one-forms ($\sigma^i, dr = \frac{x^\mu dx^\mu}{r}$) and the Hopf coordinates T^i.

shown in [7, 26] that the result (3.14) can be obtained from the commutative description of the Nekrasov-Schwarz instanton. Therefore, starting from the Eguchi-Hanson metric (3.8) in four-dimensional Euclidean gravity, we precisely derived U(1) gauge fields of the Nekrasov-Schwarz instanton and thus checked the bottom-up approach of emergent gravity [22]. This fact can be further confirmed by calculating the U(1) field strength (2.4) using the exact SW-map (2.5):

$$\widehat{F}_{\mu\nu}(x) = \frac{4}{r^2} \frac{\sqrt{1+\frac{t^4}{r^4}}-1}{\sqrt{1+\frac{t^4}{r^4}}+1} \overline{\eta}^i_{\mu\nu} T^i. \tag{3.15}$$

In the end the bottom-up approach nicely verifies the result in [26] that the Eguchi-Hanson metric (3.8) is coming from NC U(1) instanton satisfying the self-duality equation (2.21).

In section 2 we observed that the self-duality equation (2.21) for NC U(1) instantons can be written in several equivalent forms, (2.23), (2.29) and (2.35), in the commutative description after the SW map. Considering the fact that they have quite variant expressions compared to each other at first sight, the existence of such equivalent statements looks a bizarre property. In order to check the interesting identities, first note the relation (2.38) where the metric $g_{\mu\nu}$ refers to (3.10) and so the metric (2.14) is represented by

$$G_{\mu\nu} = \begin{pmatrix} G_1 & 0 & G_3 & G_4 \\ 0 & G_1 & -G_4 & G_3 \\ G_3 & -G_4 & G_2 & 0 \\ G_4 & G_3 & 0 & G_2 \end{pmatrix} \tag{3.16}$$

where

$$G_1 = P - QT^3, \qquad G_2 = P + QT^3,$$
$$G_3 = QT^1, \qquad G_4 = -QT^2$$

and

$$P = \frac{(r^2+\sqrt{r^4+t^4})^2}{4r^2\sqrt{r^4+t^4}}, \qquad Q = \frac{t^4}{2r^4\sqrt{r^4+t^4}}.$$

In a compact notation, the metric (3.16) can be written as

$$G_{\mu\nu}(x) = P\delta_{\mu\nu} + Q(\eta^3\overline{\eta}^i)_{\mu\nu}T^i. \tag{3.17}$$

The next thing is to calculate the square root of $\det G_{\mu\nu}$ which reads as

$$\sqrt{G} = G_1G_2 - (G_3^2+G_4^2) = P^2 - \frac{r^4}{4}Q^2. \tag{3.18}$$

It is now straightforward to check the self-duality equation (2.29) (with $-$-sign) using the results in (3.14) and (3.16). It is amusing to see that the symplectic U(1) gauge fields derived from the Eguchi-Hanson metric manifestly become anti-self-dual with respect to the metric (2.14) generated by themselves while they are neither self-dual nor anti-self-dual with respect to the flat metric on $\mathbb{R}^4$ as one can see from (3.14).

But note that (2.29) takes exactly the same form as the self-duality equation defined on a Riemannian manifold with the metric (2.14). Indeed we showed that (2.29) can be cast into the form (2.32) when we define $F_{ab} = E_a^\mu E_b^\nu F_{\mu\nu}$. In order to properly understand (2.32), we have to point out a caveat. So far it was not necessary to distinguish between the world (curved space) indices $\mu, \nu, \cdots$ and frame (tangent space) indices $a, b, \cdots$. (See the footnote 5.) Now, if one intends to interpret (2.29) as the form (2.32), the μ, ν indices in $F_{\mu\nu} = E_\mu^a E_\nu^b F_{ab}$ have to be regarded as the world indices and so they must be raised and lowered using the metric $G_{\mu\nu}$ in exactly the same way as in general relativity. If we adopt this interpretation, we get a remarkable picture about NC gauge fields. A naive observation is the following. The self-duality equation (2.29) says that the commutative field strengths $F_{\mu\nu}$ are (anti-)self-dual with respect to the metric $G_{\mu\nu}$. If we introduce a local basis $\{E_a\}$ for the tangent bundle TM and the dual basis $\{E^a \in T^*M\}$ defined by (2.30), the self-duality equation (2.29) can be written in the form (2.32) in a locally inertial frame where F_{ab} become (anti-)self-dual with respect to the flat metric δ_{ab} . We are already familiar with such an example being anti-self-dual with respect to the flat metric δ_{ab} and indeed (3.15) is a unique candidate satisfying such property. This reasoning implies an intriguing relation

$$F_{ab} = {E_a}^\mu F_{\mu\nu} {E^\nu}_b = \widehat{F}_{ab} \tag{3.19}$$

where $\widehat{F}_{ab}$ is given by (3.15) with the replacement $(\mu, \nu) \to (a, b)$. Now we will prove the above identity.

It is easy to find the vierbeins ${E_\mu}^a$ and the inverse vierbeins ${E_a}^\mu$ from the metric (3.17):

$$\begin{aligned} {E_\mu}^a &= C{\delta_\mu}^a + D(\eta^3 \overline{\eta}^i)_\mu{}^a T^i, && (3.20) \\ {E_a}^\mu &= \pm G^{-1/4}\big(C{\delta_a}^\mu - D(\eta^3 \overline{\eta}^i)_a{}^\mu T^i\big), && (3.21) \end{aligned}$$

where $\sqrt{G}$ is given by (3.18) and

$$C^2 = \frac{1}{2}\big(P \pm G^{1/4}\big), \qquad D^2 = \frac{2}{r^4}\big(P \mp G^{1/4}\big). \tag{3.22}$$

Here we understand the above matrix products as $(AB)_\mu{}^a = A_{\mu\lambda} B^{\lambda a}$ and $(AB)_a{}^\mu = A_{a\lambda} B^{\lambda\mu}$. We define (A.6) and (A.7) with the matrix product and used them to derive the above results. Since $G_{\mu\nu} = \delta_{\mu\nu} + (F\theta)_{\mu\nu}$ in (3.17), one can represent the U(1) field strength as

$$F_{\mu\nu} = 2\big(Q\overline{\eta}^i_{\mu\nu} T^i + (1-P)\eta^3_{\mu\nu}\big). \tag{3.23}$$

It is then straightforward to derive the fancy formula (3.19) using (3.21) and (3.23).

One can similarly understand the self-duality equation (2.35). Let us define

$$B = \frac{1}{2} B_{ab} E^a \wedge E^b = -\frac{1}{\theta}\eta^3_{ab} E^a \wedge E^b \equiv -\frac{2}{\theta}\Omega \tag{3.24}$$

where we used the relation $B_{ab} = -\frac{2}{\theta}\eta^3_{ab}$. Then the self-duality equation (2.35) is automatically satisfied since (3.24) can be written in the form (2.35) (with +-sign). That is, we

understand the background B-field as $B_{\mu\nu} \equiv -\frac{2}{\theta} E_\mu{}^a \eta^3_{ab} E^b{}_\nu$. A straightforward calculation shows that

$$\begin{aligned} B_{\mu\nu} &= 2\big(-P\eta^3_{\mu\nu} + Q\overline{\eta}^i_{\mu\nu}T^i\big) = -2\big(G\eta^3\big)_{\mu\nu} \\ &= -2\eta^3_{\mu\nu} + F_{\mu\nu} \end{aligned} \tag{3.25}$$

where we used (3.23). It is obvious that $B = \frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu$ is a closed two-form, i.e. $dB = 0$ as long as $dF = 0$–the Bianchi identity. Then (3.24) implies that Ω is the Kähler form of the metric (3.16) and (1,1)-form with respect to the complex structure $J^a{}_b = \eta^3_{ab}$ [27, 29]. In conclusion, we got a very nice interpretation of the self-duality equations (2.29) and (2.35) consistent with general relativity.

It is straightforward to generalize the bottom-up approach to a space with the metric (3.1). After a little algebra we find that the metric (3.1) can be written as

$$g_{\mu\nu}(x) = \frac{1}{2}(A^2+B^2)\delta_{\mu\nu} - \frac{1}{r^2}(A^2-B^2)(\eta^3\overline{\eta}^i)_{\mu\nu}T^i \tag{3.26}$$

and so the U(1) field strength in (3.13) is given by

$$F_{\mu\nu}(x) = f_1(r)\eta^3_{\mu\nu} + f_2(r)\overline{\eta}^i_{\mu\nu}T^i \tag{3.27}$$

where

$$f_1(r) = 1 - \frac{1}{2}(A^2+B^2) \qquad f_2(r) = -\frac{1}{r^2}(A^2-B^2). \tag{3.28}$$

Using the result (3.27) one can also calculate the inverse metric

$$\begin{aligned} G^{\mu\nu} &= \Big(\frac{1}{1+F\theta}\Big)^{\mu\nu} \\ &= \frac{2+A^2+B^2}{1+A^2+B^2+A^2B^2}\delta_{\mu\nu} + \frac{2(A^2-B^2)}{r^2(1+A^2+B^2+A^2B^2)}\big(\eta^3\overline{\eta}^i\big)_{\mu\nu}T^i \end{aligned} \tag{3.29}$$

and the field strength (2.4) of symplectic U(1) gauge fields

$$\widehat{F}_{\mu\nu}(x) = \frac{2}{1+A^2+B^2+A^2B^2}\Big[(1-A^2B^2)\eta^3_{\mu\nu} - \frac{2}{r^2}(A^2-B^2)\overline{\eta}^i_{\mu\nu}T^i\Big]. \tag{3.30}$$

It is now obvious that NC U(1) instantons correspond to the metric (3.1) with $A^2B^2 = 1$ that is precisely the case (3.9) for the Eguchi-Hanson metric.

A geometrical meaning of the 't Hooft symbols defined in appendix A is to specify the triple (I, J, K) of complex structures of $\mathbb{R}^4$ for a given orientation. Therefore the choice of a particular NC parameter in (2.25), e.g. $\theta^{\mu\nu} = \frac{\theta}{2}\eta^3_{\mu\nu}$, corresponds to singling out a particular complex structure, for example, $J = T^3_+$ in (A.10). And the space (3.1) inherits the complex structure J from $\mathbb{R}^4$. So let us consider the fundamental 2-form defined by

$$\omega = \frac{1}{2}\eta^3_{ab}e^a \wedge e^b. \tag{3.31}$$

Now we will show that the fundamental 2-form ω is closed, i.e. $d\omega = 0$, and so defines the Kähler form of the metric (3.1) as far as the U(1) field strength (3.27) obeys the Bianchi

identity, i.e. $dF = 0$. In other words, the metric (3.1) is always Kähler if and only if $dF = 0$. Using the covectors in (3.2) and coframes in (3.3), the 2-form ω can be written as

$$\omega = \frac{1}{2}\Big[\frac{1}{2}(A^2 + B^2)\eta^3_{\mu\nu} + \frac{1}{r^2}(A^2 - B^2)\overline{\eta}^i_{\mu\nu}T^i\Big]dx^\mu \wedge dx^\nu \tag{3.32}$$

where we used the identity (3.12) and (A.8). After using the relation (3.28), the 2-form in (3.32) finally reduces to

$$\omega = \frac{1}{2}\eta^3_{\mu\nu}dx^\mu \wedge dx^\nu - \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu = \omega^{(0)} - F \tag{3.33}$$

where $\omega^{(0)} = \frac{1}{2}\eta^3_{\mu\nu}dx^\mu \wedge dx^\nu$. Consequently, $d\omega = 0$ if and only if $dF = 0$.[6]

One may wonder whether one can get U(1) gauge fields in the same way from the Taub-NUT metric [41] which takes the form

$$ds^2 = \frac{1}{4}\frac{\rho+m}{\rho-m}d\rho^2 + (\rho^2 - m^2)(\sigma_1^2 + \sigma_2^2) + 4m^2\frac{\rho-m}{\rho+m}\sigma_3^2. \tag{3.34}$$

A critical difference from the Eguchi-Hanson metric (3.8) is that the Taub-NUT metric (3.34) is locally asymptotic at infinity to $\mathbb{R}^3 \times \mathbb{S}^1$ and so it belongs to the class of asymptotically locally flat (ALF) spaces. Therefore the Taub-NUT metric cannot be represented by the Hopf coordinates (3.5) and it is difficult to naively generalize the previous construction to ALF spaces. From the gauge theory point of view, it may be related to the fact that ALF spaces arise from NC monopoles [43] whose underlying equation is defined by an $\mathbb{S}^1$-compactification of (2.21), the so-called Nahm equation. We will discuss in [44] a possible generalization to include the Taub-NUT metric (3.34) in the bottom-up approach of emergent gravity.

3.2 Gravitational metric from Braden-Nekrasov instanton

Braden and Nekrasov [28] considered a deformed ADHM construction on *commutative* $\mathbb{C}^2$ whose solution gives a resolved moduli space $\widetilde{\mathcal{M}}_{N,k} = \mu^{-1}(\vec{\zeta})/U(k)$ in terms of hyper-Kähler quotient where $\mu^{-1}(\vec{\zeta})$ are $U(k)$ hyper-Kähler moment maps [45, 46]. It was shown in [25] that the same resolved instanton moduli space arises by considering the standard (undeformed) ADHM construction but instead assuming the spacetime coordinates having the commutation relation (1.1) where the deformation parameters $\vec{\zeta}$ in the $U(k)$ hyper-Kähler moment maps $\mu^{-1}(\vec{\zeta})$ then arise from $\theta^{(\pm)i}$ in (2.25). Thus it will be interesting to study the relation between the deformed ADHM construction of an ordinary commutative gauge theory and the undeformed ADHM construction of a NC gauge theory. Furthermore, as we discussed in section 1, the NC gauge theory can be mapped to the ordinary commutative gauge theory by the SW map. Therefore one may expect that NC U(1) instantons constructed in [25] would be related by the SW map to U(1) instanton solutions constructed by the deformed ADHM data on commutative $\mathbb{C}^2$. Interestingly the commutative U(1) instantons seem not to be related to the NC U(1) instantons by the SW map

[6] One can easily see that the metric (3.26) can be written as $g_{\mu\nu}(x) = -\omega_{\mu\lambda}(x)\eta^3_{\lambda\nu}$ where $\omega_{\mu\nu}(x)$ is defined by (3.32). This is nothing but the definition of Kähler form with the complex structure $J^\mu{}_\nu = \eta^3_{\mu\nu}$: $\omega(X,Y) = g(X,JY)$ for vector fields $X, Y \in TM$.

as already noted in [28] (see *Notes added five years later* in section 6.2) and the commutative description of the Nekarasov-Schwarz (NS) instanton is indeed different from the Braden-Nekrasov (BN) U(1) instanton as was shown in [47].

We want to shed light on this puzzle by studying the geometrical properties of a four-manifold determined by the BN U(1) instanton in the context of emergent gravity and by comparing the result to the NS instanton whose SW map gives rise to a complete regular geometry described by the Eguchi-Hanson metric as was shown in section 3.1. But it seems necessary to consider a full NC deformation to completely understand the topology change of spacetime and a subsequent resolution of spacetime singularities and and so it still waits for a complete explanation.

It turns out [28] that U(1) instantons constructed from the deformed ADHM construction on commutative $\mathbb{C}^2$ are still singular unlike NC U(1) instantons and so it is necessary to change the topology of spacetime in order to make the corresponding U(1) gauge fields non-singular. The reason that an Abelian instanton exists is that spacetime is now blown up and there are non-contractible 2-spheres. Then the resulting spacetime is not $\mathbb{C}^2$ but a Kähler manifold X which is a blowup of $\mathbb{C}^2$ at a finite number of points. In the end U(1) gauge fields on X are well-defined and carry a nontrivial second Chern class k as well as a nontrivial first Chern class when the gauge fields are restricted to exceptional divisors. But, unfortunately, the blowup becomes manifest only by gluing local coordinate patches and performing a proper gauge transformation on their intersections. For example, one can choose local coordinates (t, λ) on a patch $\mathcal{U}_0$ such that $z_1 = t,\ z_2 = t\lambda$ where $(z_1, z_2) \in \mathbb{C}^2$ and another local coordinates (s, μ) on another patch $\mathcal{U}_\infty$ such that $z_1 = \mu s,\ z_2 = s$. On these patches $\mathcal{U}_0$ and $\mathcal{U}_\infty$, the point $0 = (0, 0)$ in $\mathbb{C}^2$ is replaced by the space $\mathbb{CP}^1$ of complex lines passing through the point 0. One can use the local coordinates to represent U(1) gauge fields on each local chart and then extend them via a gauge transformation to a safe region where $(z_1, z_2) \neq 0$ [28].

Now let us undertake a more systematic investigation of the charge one U(1) instanton in the section 4 of [28]. The instanton gauge fields are given by (setting $\theta = 1$)

$$A = \frac{1}{2r^2(1+r^2)}(z_1 d\overline{z}_1 - \overline{z}_1 dz_1 + z_2 d\overline{z}_2 - \overline{z}_2 dz_2) \tag{3.35}$$

and

$$F = \frac{dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2}{r^2(1+r^2)} - \frac{1+2r^2}{r^4(1+r^2)^2} \sum_{i,j} z_i \overline{z}_j dz_j \wedge d\overline{z}_i \tag{3.36}$$

where $z_1 = x^1 + ix^2,\ z_2 = x^3 + ix^4$ and $r^2 = |z_1|^2 + |z_2|^2$. The above gauge fields $A \equiv 2iA_\mu dx^\mu$ (the factor 2 scaling is just for convenience) and $F = dA = iF_{\mu\nu} dx^\mu \wedge dx^\nu$ can be represented in terms of the Cartesian coordinates $\{x^\mu\}$ and can be written as

$$A_\mu(x) = \frac{t^2}{2r^2(t^2+r^2)} \eta^3_{\mu\nu} x^\nu \tag{3.37}$$

and

$$F_{\mu\nu}(x) = -\frac{t^4}{2r^2(t^2+r^2)^2} \eta^3_{\mu\nu} + \frac{t^2(t^2+2r^2)}{r^4(t^2+r^2)^2} \overline{\eta}^i_{\mu\nu} T^i, \tag{3.38}$$

where we recovered the dimensionful parameter $t^2 = \theta$ for an accessible comparison with the NS instanton. To get the expression (3.38), we used the identity (3.12). It may be interesting to compare the asymptotic behaviors (set $t = 1$) of the NS instanton (3.14) and the BN instanton (3.38) both in $r \to \infty$

$$\begin{aligned}
\mathrm{NS}: \ F_{\mu\nu}(x) &= \frac{1}{r^6}\Big(1 - \frac{1}{2r^4} + \cdots\Big)\overline{\eta}^i_{\mu\nu}T^i - \frac{1}{8r^8}\Big(1 - \frac{1}{r^4} + \cdots\Big)\eta^3_{\mu\nu}, \\
\mathrm{BN}: \ F_{\mu\nu}(x) &= \frac{2}{r^6}\Big(1 - \frac{3}{2r^2} + \cdots\Big)\overline{\eta}^i_{\mu\nu}T^i - \frac{1}{2r^6}\Big(1 - \frac{2}{r^2} + \cdots\Big)\eta^3_{\mu\nu},
\end{aligned} \tag{3.39}$$

and in $r \to 0$

$$\begin{aligned}
\mathrm{NS}: \ F_{\mu\nu}(x) &= \frac{1}{r^4}\Big(1 - \frac{1}{2}r^4 + \frac{3}{8}r^8 + \cdots\Big)\overline{\eta}^i_{\mu\nu}T^i - \frac{1}{2r^2}\Big(1 - 2r^2 + \frac{3}{2}r^4 + \cdots\Big)\eta^3_{\mu\nu}, \\
\mathrm{BN}: \ F_{\mu\nu}(x) &= \frac{1}{r^4}\Big(1 - r^4 + 2r^6 + \cdots\Big)\overline{\eta}^i_{\mu\nu}T^i - \frac{1}{2r^2}\Big(1 - 2r^2 + 3r^4 + \cdots\Big)\eta^3_{\mu\nu}.
\end{aligned} \tag{3.40}$$

One can see [28] that the asymptotic behaviors for two instantons are almost the same except that the BN instanton is slightly slowly decaying at $r \to \infty$.

Note that the instanton gauge field (3.37) was obtained through the ADHM construction. Nevertheless, its field strength (3.38) is neither self-dual nor anti-self-dual. We remark that this fact was already observed in Eq. (3.14) in [28]. A notable point is that the commutative description of the NS instantons also shares this feature as shown in (3.14). But (3.15) verifies that the NS instanton becomes (anti-)self-dual in the NC description. Thus one may wonder whether the same property can be realized even for the BN instantons. To see what happens in the NC description of the BN instantons, let us apply the SW map (2.5) to the U(1) field strength (3.38). The result is given by (3.30) with

$$A^2(r) = \frac{r^2(r^2 + 2t^2)}{(r^2 + t^2)^2}, \qquad B^2(r) = \frac{r^4 + r^2t^2 + t^4}{r^2(r^2 + t^2)}. \tag{3.41}$$

Explicitly it takes the form

$$\widehat{F}_{\mu\nu}(x) = \frac{2t^2}{(2r^4 + 2r^2t^2 + t^4)(2r^4 + 4r^2t^2 + t^4)}\Big[\frac{2(r^2 + t^2)(2r^2 + t^2)}{r^2}\overline{\eta}^i_{\mu\nu}T^i - r^2t^2\eta^3_{\mu\nu}\Big]. \tag{3.42}$$

The result (3.42) shows that the BN instanton is neither self-dual nor anti-self-dual even in the NC description. It can be understood as follows. First note that the field strength (3.38) takes the form (3.27) with the coefficients (3.28) and the resultant NC field strength is then given by (3.30) where A and B are given by (3.41). But (3.38) does not satisfy the relation $A^2B^2 = 1$, which leads to the former conclusion. This presents a sharp contrast with the NS instantons with (anti-)self-dual curvatures in NC spacetime.

One may wonder whether the identity (3.19) is true even for the BN instanton. That is, one can ask whether the NC field strength (3.42) can be written as $\widehat{F}_{ab} = E_a{}^\mu F_{\mu\nu} E^\nu{}_b$ with the commutative field strength (3.38) where the vierbeins are defined by $G_{\mu\nu} = \delta_{\mu\nu} + (F\theta)_{\mu\nu} = E^a_\mu E^a_\nu$. We checked that the identity (3.19) still holds for the BN instanton which is neither self-dual nor anti-self-dual. The proof goes through the same way as the NS instanton case. We will present in section 5 a proof of the identity (3.19) for general U(1) gauge fields with the symmetric metric (2.14).

We analyzed before the asymptotic behavior of the BN instanton and found that the leading behavior is exactly the same as the NS instanton. An interesting question is then whether a four-dimensional manifold determined by the BN instanton also exhibits a similar geometrical countenance. In order to investigate the geometrical properties of the four-manifold, let us consider the metric $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = e^a \otimes e^a$ defined by (3.13) with the solution (3.38). But we want to express the metric in the form (3.1) using the left-invariant one-forms in (3.3). To implement this form, one can employ the reverse procedure of section 3.1 to arrive at the result

$$ds^2 = \frac{r^2(r^2+2t^2)}{(r^2+t^2)^2}(dr^2 + r^2\sigma_3^2) + \frac{r^4+r^2t^2+t^4}{r^2+t^2}(\sigma_1^2+\sigma_2^2) \tag{3.43}$$

and so A^2 and B^2 are given by (3.41). This metric form would indicate that the four-manifold described by (3.43) might be akin to the Eguchi-Hanson metric (3.8). For example, the metric (3.43) also contains a nontrivial two-cycle $\mathbb{S}^2$ at the origin ($r=0$) where the metric is degenerated to the two dimensional sphere with the metric $t^2(\sigma_1^2+\sigma_2^2)$. Probably, the two-sphere in (3.43) is related to the Kähler blowup for the BN instanton at the origin of $\mathbb{C}^2$. In order to understand the engrossing feature, let us recapitulate a corresponding aspect for the NS instanton [8]. The Eguchi-Hanson metric (3.8) has a curvature that reaches a maximum at the 'origin' $\rho = t$ (recall that $\rho^4 = r^4 + t^4$), falling away to zero in all four directions as the radius ρ increases. Since the radial coordinate runs down only as far as $\rho = t$, there is a minimal 2-sphere $\mathbb{S}^2$ of radius t described by the metric $t^2(\sigma_1^2+\sigma_2^2)$. This degeneration of the generic three dimensional orbits to the two dimensional sphere is known as a 'bolt' [48]. But, $\rho = t$ corresponds to the origin $r = 0$ of the embedding coordinates in field theory and so this nontrivial topology is not visible in the gauge theory description. However, as we showed in section 3.1, the emergent gravity approach where a Riemannian manifold is emerging from dynamical gauge fields reveals a nontrivial topology of NC U(1) gauge fields. As one can see from (3.13), if $F = 0$, the corresponding spacetime is $\mathbb{R}^4$ without any nontrivial cycles but, if the intanton gauge fields in (3.14) are developed, the spacetime evolves to the Eguchi-Hanson space which contains a non-contractible 2-sphere dubbed as the bolt. Therefore the emergent gravity verifies the topology change of spacetime due to U(1) instantons. Exactly the same phenomenon happened for the BN instanton. But more detailed analysis brings some surprise.

It is straightforward to calculate the spin connections and curvature tensors of the metric (3.43) using the results of appendix B. We present the explicit result for reader's convenience:

$$\begin{aligned}
\omega_{12} &= -\frac{r^6+2r^4t^2+4r^2t^4+2t^6}{r^2\sqrt{r^2+2t^2}(r^4+r^2t^2+t^4)}e^3, &\qquad \omega_{34} &= \frac{r^4+3r^2t^2+4t^4}{r^2(r^2+2t^2)^{3/2}}e^3, \\
\omega_{13} = -\omega_{42} &= \frac{r^2\sqrt{r^2+2t^2}}{r^4+r^2t^2+t^4}e^2, &\qquad \omega_{14} = -\omega_{23} &= \frac{r^2\sqrt{r^2+2t^2}}{r^4+r^2t^2+t^4}e^1,
\end{aligned} \tag{3.44}$$

and

$$\begin{aligned}
R_{12} &= Xe^1\wedge e^2 - 2Ye^3\wedge e^4, &\qquad R_{34} &= -2Ye^1\wedge e^2 + Ze^3\wedge e^4, \\
R_{13} = -R_{42} &= Y(e^3\wedge e^1 - e^2\wedge e^4), &\qquad R_{14} = -R_{23} &= Y(e^2\wedge e^3 - e^1\wedge e^4),
\end{aligned} \tag{3.45}$$

where the vierbeins e^a are given by (3.2) with the result (3.41) and X, Y, Z are given by

$$\begin{aligned} X &= \frac{4t^4(2r^2+t^2)}{(r^4+r^2t^2+t^4)^2}, \\ Y &= \frac{t^4(4r^4+7r^2t^2+4t^4)}{(r^2+2t^2)(r^4+r^2t^2+t^4)^2}, \\ Z &= \frac{4t^4(2r^2+3t^2)}{r^2(r^2+2t^2)^3}. \end{aligned} \tag{3.46}$$

Using the result (3.45), one can easily read off the Ricci tensor $R_{ab} \equiv R_{acbc}$ and the Ricci scalar $R \equiv R_{aa}$. We present only the diagonal Ricci tensors which read as

$$\begin{aligned} R_{11} = R_{22} &= \frac{6r^2t^6}{(r^2+2t^2)(r^4+r^2t^2+t^4)^2}, \\ R_{33} = R_{44} &= -\frac{6t^6(3r^8+8r^6t^2+6r^4t^4-2t^8)}{r^2(r^2+2t^2)^3(r^4+r^2t^2+t^4)^2}. \end{aligned} \tag{3.47}$$

Therefore the Ricci scalar is given by

$$R = -\frac{24t^6(r^4+r^2t^2-t^4)}{r^2(r^2+2t^2)^3(r^4+r^2t^2+t^4)}. \tag{3.48}$$

Some remarks are in order. We proved in (3.33) that the metric (3.1) becomes Kähler if the U(1) field strength (3.27) derived from the metric satisfies the Bianchi identity $dF = 0$. Note that the metric (3.43) was derived from the U(1) field strength (3.36) which satisfies the Bianchi identity $dF = 0$. Therefore the metric (3.43) must always be Kähler. If one looks at the curvature tensors in (3.45), one can see that the four-manifold generated by the BN instanton is neither self-dual nor anti-self-dual though it is close to an anti-self-dual manifold. This is consistent with the gauge theory result. Moreover it has a nontrivial Ricci scalar which is divergent at the origin. This indicates that the classical geometry emergent from the BN instanton contains a spacetime singularity at the origin. To see that this is a true singularity, one must look at quantities that are independent of the choice of coordinates. An obvious candidate is of course the Ricci scalar R which is already singular in our case. But it may identically vanish for some solutions, for example, the Schwarzschild black hole. For such cases, another important quantity is the Kretschmann invariant which is defined by $K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. The existence of the singularity can be verified by noting that either the Ricci scalar R or the Kretschmann scalar K is infinite. For example, the famous Schwarzschild black hole exhibits such a spacetime singularity for which the Kretschmann scalar K is given by

$$K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6} \tag{3.49}$$

which blows up at $r = 0$ indicating the presence of a singularity. Thus one can calculate the Kretschmann scalar K for the metric (3.43) in order to further confirm the spacetime singularity which is given by

$$\frac{K}{64t^8} = \frac{(2r^2+3t^2)^2}{r^4(r^2+2t^2)^6} + \frac{(2r^2+t^2)^2}{(r^4+r^2t^2+t^4)^4} + \frac{(4r^4+7r^2t^2+4t^4)^2}{(r^2+2t^2)^2(r^4+r^2t^2+t^4)^4}. \tag{3.50}$$

One can clearly see that the first term becomes divergent at the origin although next the two terms are regular everywhere as long as $t^2 = \theta \neq 0$. And so (3.50) again verifies the spacetime singularity at the origin.

Let us try to understand why the classical geometries generated by the BN instanton and the NS instanton are so dissimilar, especially, in view of the singularity structure. From the gauge theory point of view, both instantons are constructed by solving the ADHM data for the rank one gauge group. The hyper-Kähler moment maps $\mu^{-1}(\vec{\zeta})$ are deformed, i.e. $\vec{\zeta} \neq 0$, for both cases. But the origin is different. For the BN instanton case, the non-vanishing deformation parameters $\vec{\zeta}$ are assumed from the beginning in the ADHM data which are defined on *commutative* $\mathbb{C}^2$. But, for the NS instanton case, the non-vanishing deformation parameters $\vec{\zeta}$ are not introduced by hand. Instead the ADHM data are now defined on *noncommutative* $\mathbb{C}^2$ obeying the Heisenberg algebra

$$[z_i, z_j^\dagger] = \zeta_{\mathbb{R}} \delta_{ij}, \qquad i, j = 1, 2, \tag{3.51}$$

where $\zeta_{\mathbb{R}} = \frac{1}{2}\eta^3_{\mu\nu}\theta^{\mu\nu}$. The deformation parameter $\zeta_{\mathbb{R}}$ appears in the moment maps $\mu^{-1}(\vec{\zeta})$ due to the NC algebra (3.51) where the remaining deformation parameters $\zeta_{\mathbb{C}} = \frac{1}{2}(\eta^1_{\mu\nu} + i\eta^2_{\mu\nu})\theta^{\mu\nu}$ may be nullified by performing an SO(4) rotation. An important condition in the ADHM construction is the completeness relation which ensures a canonical decomposition of a vector space $\mathbb{C}^{N+2k}$ into some null-space annihilated by a Dirac operator $\mathcal{D}^\dagger : \mathbb{C}^{N+2k} \to \mathbb{C}^{2k}$ and its orthogonal complement. If the completeness relation is obeyed, the ADHM gauge fields are necessarily self-dual or anti-self-dual [45]. A magical upshot of the ADHM construction is that the completeness relation is well-defined even in NC space as was shown in [36, 37, 38]. (See also section 4 in [39].) Thus the NC instantons constructed via the ADHM construction must be self-dual or anti-self-dual. Furthermore the NC space (3.51) resolves the singularities of instanton moduli space coming from point-like instantons which shrink to zero size [25]. Therefore the NC instantons are well-defined and nonsingular. However, this feature is lacking if the deformed ADHM data are defined over a commutative space. The completeness relation fails at a finite number of points, called "freckles." As a result, the ADHM gauge fields are no longer (anti-)self-dual as one can see from (3.38) (see also Eq. (3.14) in [28]). The spacetime singularity in (3.48) and (3.50) arises at the freckle where the instanton is placed.

In the gauge theory description, the gauge fields of BN instantons become nonsingular after a Kähler blow up at a finite number of points on $\mathbb{C}^2$ which brings about the topology change of spacetime [28]. But the topology change is rather mysterious if we recall that it is purely a gauge theory description. Nevertheless, as we observed in (3.43), the emergent gravity description provides an amicable picture that the topology change may be understood by describing U(1) gauge fields as dynamical manifolds. Actually this possibility was suspected in [28] (see section 6.2). But life is not simple. It is difficult to concretely demonstrate the topology change unlike the NS instanton case because the metric (3.43) for the BN instanton holds a spacetime singularity at the blow up point. However we emphasized in section 1 that the emergent gravity picture can be realized only by turning on B-fields to admit a symplectic structure and allowing U(1) gauge fields to deform the symplectic structure. Then the presence of B-fields will necessarily give rise to the NC

space (1.1), especially, near spacetime singularities. So, according to the quantization map (1.11), the commutative Poisson algebra $\mathfrak{P} = (C^\infty(M), \{-,-\}_\theta)$ will be mapped to the NC algebra $(\mathcal{A}_\theta, \mathcal{H}, [-,-])$. In this case U(1) instantons are defined on the NC space (1.1) which is precisely the setup of NS instantons. And we showed that the NS instanton has the commutative description by the SW map in terms of the Eguchi-Hanson space which is a regular geometry without any spacetime singularity. This shows a sharp contrast between the NS instanton and the BN instanton. This argument implies that, in order to describe the topology change of spacetime and the resolution of spacetime singularity in any field theory, it is not enough to deform only the ADHM data leaving spacetime to be commutative. The NC structure of spacetime may be essential to resolve the spacetime singularities in general relativity. Therefore it is necessary to take the NC space (1.1) at the outset and then consider the commutative description of NC gauge theory.

4. Topological invariants of U(1) gauge fields

The emergent gravity raises an intriguing question about topological invariants in gravity and U(1) gauge theory. In the gravity side, there are two topological invariants associated with the Atiyah-Patodi-Singer index theorem for an elliptic complex in four dimensions [41], namely the Euler characteristic $\chi(M)$ and the Hirzebruch signature $\tau(M)$, which can be expressed as integrals of the curvature of a four dimensional metric while there is no natural topological invariant in the U(1) gauge theory. For example, the second Chern class of U(1) bundle on $\mathbb{R}^4$ is trivial and so it is easy to show that a non-trivial instanton charge is incompatible with the vanishing of $F = dA$ at infinity. The second Chern class of U(1) bundle can be well-defined only for NC U(1) instantons [49, 39, 40]. But it was proved [2, 29] that the commutative limit of NC U(1) instantons is equivalent to gravitational instantons. Hence the emergent gravity implies that the commutative limit of NC U(1) gauge fields has to carry the same topological invariants as four-dimensional Riemannian manifolds. Thus a natural question is how to interpret the two topological invariants of four-manifolds in terms of U(1) gauge fields in the context of emergent gravity.

First it will be interesting to compare the instanton number for the NS and BN instantons. Using the results (3.14) and (3.38), one can get

$$\text{NS}: \quad F \wedge F = -\frac{2}{r^2\sqrt{r^4+t^4}}\big(\sqrt{r^4+t^4} - r^2\big)^2 d^4x, \tag{4.1}$$

$$\text{BN}: \quad F \wedge F = -\frac{8t^2}{r^2(r^2+t^2)^3} d^4x \tag{4.2}$$

and so the instanton number is given by[7]

$$\text{NS}: \quad I = \frac{1}{4\pi^2}\int F \wedge F = -\frac{t^4}{4}, \tag{4.3}$$

$$\text{BN}: \quad I = \frac{1}{4\pi^2}\int F \wedge F = -1. \tag{4.4}$$

[7] Here we are considering the anti-self-dual instantons with real F and adopt the normalization $1/4\pi^2$ in [28] for the instanton number I which is different from $1/8\pi^2$ in (4.5). Since $F \wedge F = d(A \wedge F)$ and $F \to 0$ as $r \to \infty$, it is obvious that the contribution to the instanton number I is localized at the origin.

It is amusing to notice that the instanton number for the NS instanton depends on $t^4 = \theta^2$ while the BN instanton does not. Actually this fact for the former case was observed in [7, 26] and was interpreted as a nonperturbative breakdown of the SW map due to a finite radius of convergence [7, 9]. But the BN instanton solution (3.35) was directly obtained by solving the ADHM equations where the instanton number $k = |I|$ specifies the dimension of the vector space $\mathbb{C}^{N+2k}$. Thus k should be an integer number for consistency. For the same reason, the instanton number for NC U(1) instantons satisfying the self-duality equation (2.21) must be an integer $k \in \mathbb{Z}$ [49, 39, 40] which is defined by

$$I = \frac{1}{8\pi^2} \int \widehat{F} \wedge \widehat{F} \in \mathbb{Z} \tag{4.5}$$

where $\widehat{F} = d\widehat{A} - i\widehat{A} \wedge \widehat{A}$. The NS instanton case is rather puzzling because the instanton number is not quantized and so the topology of U(1) gauge fields becomes obscure. But, as we argued above, the commutative limit of NC U(1) gauge fields carries the topological information in the form of four-dimensional Riemannian manifolds. In a deep NC space where the continuum description in terms of smooth geometries becomes bad, the NC U(1) gauge bundle whose invariant is given by (4.5) will take over the topological information.

Now we will investigate with explicit examples in section 3 how the topological information of U(1) gauge fields is reflected in a four-dimensional Riemannian manifold. The topological invariants for four-manifolds have a local expression due to the Atiyah-Singer index theorem [41, 42]. For a general Riemannian manifold M, the Euler number $\chi(M)$ for the de Rham complex and the signature $\tau(M)$ for the Hirzebruch signature complex are defined by

$$\begin{aligned} \chi(M) &= \frac{1}{32\pi^2} \int_M \varepsilon^{abcd} R_{ab} \wedge R_{cd} \\ &\quad + \frac{1}{16\pi^2} \int_{\partial M} \varepsilon^{abcd} \Big(v_{ab} \wedge R_{cd} - \frac{2}{3} v_{ab} \wedge v_{ce} \wedge v_{ed} \Big), \qquad (4.6) \\ \tau(M) &= -\frac{1}{24\pi^2} \int_M \mathrm{Tr} R \wedge R - \frac{1}{24\pi^2} \int_{\partial M} \mathrm{Tr} v \wedge R + \eta_S(\partial M), \qquad (4.7) \end{aligned}$$

where v_{ab} is the second fundamental form of the boundary ∂M. It is defined by

$$v_{ab} = \omega_{ab} - \omega_{0ab}, \tag{4.8}$$

where ω_{ab} are the actual connection 1-forms and ω_{0ab} are the connection 1-forms if the metric were locally a product form near the boundary [41]. The connection 1-form ω_{0ab} will have only tangential components on ∂M and so the second fundamental form v_{ab} will have only normal components on ∂M. And $\eta_S(\partial M)$ is the η-function given by the eigenvalues of a signature operator defined over ∂M and depends only on the metric on ∂M [41]. The topological invariants are also related to nuts (isolated points) and bolts (two surfaces), which are the fixed points of the action of one parameter isometry groups of gravitational instantons [47]. Using the gauge theory formulation [50, 51, 52] that Einstein gravity can be formulated as a gauge theory of Lorentz group $SO(4) = SU(2)_L \times SU(2)_R$ where spin connections play the role of gauge fields and Riemann curvature tensors correspond to their

field strengths, it is possible to express the topological invariants in terms of $SU(2)_L$ and $SU(2)_R$ gauge fields

$$\begin{aligned}
\chi(M) =& \frac{1}{4\pi^2}\int_M \left(F^{(+)i}\wedge F^{(+)i} - F^{(-)i}\wedge F^{(-)i}\right) \\
&+\frac{1}{4\pi^2}\int_{\partial M}\left(a^{(+)i}-a^{(-)i}\right)\wedge\left(F^{(+)i}+F^{(-)i}\right) \\
&+\frac{1}{12\pi^2}\int_{\partial M}\varepsilon^{ijk}\left(a^{(+)i}-a^{(-)i}\right)\wedge\left(a^{(+)j}-a^{(-)j}\right)\wedge\left(a^{(+)k}-a^{(-)k}\right), \quad (4.9)\\
\tau(M) =& \frac{1}{6\pi^2}\int_M \left(F^{(+)i}\wedge F^{(+)i} + F^{(-)i}\wedge F^{(-)i}\right) \\
&+\frac{1}{12\pi^2}\int_{\partial M}\left(a^{(+)i}-a^{(-)i}\right)\wedge\left(F^{(+)i}-F^{(-)i}\right)+\eta_S(\partial M), \quad (4.10)
\end{aligned}$$

where the fundamental 1-form (4.8) is decomposed according to the Lie algebra splitting $SO(4) = SU(2)_L \oplus SU(2)_R$ as

$$v_{ab} \equiv a^{(+)i}\eta^i_{ab} + a^{(-)i}\bar{\eta}^i_{ab} \quad (4.11)$$

and the volume forms are defined by $e^1\wedge e^2\wedge e^3\wedge e^4 \equiv \sqrt{g}d^4x$ and $e^1\wedge e^2\wedge e^3|_{\partial M} \equiv \sqrt{h}d^3x$.

We showed in section 3 that U(1) gauge fields for the NS and BN instantons can be written in the form (3.27). Accordingly, given U(1) gauge fields of the form (3.27), one can calculate the gravitational metric that is given by (3.26) or equivalently (3.1). Then, using the results of appendix B, it is straightforward to calculate the Euler density $\rho_\chi(M)$ in (B.5). The results for the NS and BN instantons are, respectively, given by

$$\text{NS}: \quad \rho_\chi(M) = -\frac{24t^8}{(r^4+t^4)^3}, \quad (4.12)$$

$$\text{BN}: \quad \rho_\chi(M) = \frac{16t^8(4r^4+8r^2t^2+3t^4)}{r^2(r^2+2t^2)^3(r^4+r^2t^2+t^4)^2} + \frac{8t^8(4r^4+7r^2t^2+4t^4)^2}{(r^2+2t^2)^2(r^4+r^2t^2+t^4)^4}. \quad (4.13)$$

As expected, the Euler density for the NS instanton is regular everywhere while the Euler density for the BN instanton is singular at the origin due to the first term in (4.13). But the remarkable fact is that the singular behavior of the Euler density is much milder, i.e. $1/r^2$, than the Kretschmann scalar (3.50) which is $1/r^4$. Actually the $(1/r^2)$-singularity in the Euler density (4.13) is not harmful when we calculate the Euler characteristic (4.6) because the volume factor $\sqrt{g}d^4x$ will safely cancel the singularity. Indeed we will get a finite bulk contribution for the Euler characteristic $\chi(M)$ of the BN instanton. To be specific, the bulk part of the Euler characteristic $\chi(M)$ is given by

$$\chi_{bulk}(M) = \frac{1}{2\pi^2}\int_M \rho_\chi(M) e^1\wedge e^2\wedge e^3\wedge e^4 \quad (4.14)$$

and we get the same value $\chi_{bulk}(M) = \frac{3}{2}$ for both the cases. Here we used the fact that both the NS instanton and the BN instanton satisfy the ALE boundary condition $\mathbb{R}^4/\mathbb{Z}_2$ because they share the same asymptotic behaviors as was shown in (3.39) and (3.40) and so $\int_{\mathbb{RP}^3}\sigma^1\wedge\sigma^2\wedge\sigma^3 = \pi^2$. Now let us look at the boundary terms in (4.6). The first boundary

term will not contribute because the curvature tensor will rapidly vanish ($\sim t^4/r^6$) at infinity. But the second boundary term will contribute and the explicit computation [50] gives the value

$$\chi_{boundary}(M) = \frac{1}{2\pi^2}\int_{\mathbb{RP}^3} \sigma^1 \wedge \sigma^2 \wedge \sigma^3 = \frac{1}{2}. \tag{4.15}$$

Note that the boundary contribution for the NS and BN instantons is also the same because they satisfy the same asymptotic boundary condition. In the end we get the Euler characteristic (4.6) given by

$$\chi(M) = \frac{3}{2} + \frac{1}{2} = 2 \tag{4.16}$$

for both instantons.

The Hirzebruch signature $\tau(M)$ can be calculated similarly using the result in (B.6). But it is not necessary to separately calculate the bulk part of the Hirzebruch signature $\tau(M)$ for the NS instanton. One can easily check that the spin connections in (B.2) for the solution (3.15) are anti-self-dual, i.e., $\omega_{ab} = -\frac{1}{2}\varepsilon_{ab}{}^{cd}\omega_{cd}$ which automatically leads to anti-self-dual curvature tensors. It should be the case because the NS instanton is equivalent to the Eguchi-Hanson space which is an ALE gravitational instanton. This means that $F^{(+)i} = 0$, $\forall i$ in (4.9) and (4.10) and so the relation $\rho_\tau(M) = -\frac{2}{3}\rho_\chi(M)$ is deduced. For the BN instantons, one can directly calculate $\rho_\tau(M)$ in (B.6) using the result (3.41) or (3.45). The result can be summarized as follows:

$$\text{NS}: \ \ \rho_\tau(M) = -\frac{2}{3}\rho_\chi(M) = \frac{16t^8}{(r^4+t^4)^3}, \tag{4.17}$$

$$\begin{aligned}\text{BN}: \ \ \rho_\tau(M) = {} & \frac{2t^8(r^4+2r^2t^2+2t^4)(4r^4+7r^2t^2+4t^4)(4r^6+9r^4t^2+6r^2t^4+2t^6)}{r^2(r^2+2t^2)^4(r^4+r^2t^2+t^4)^4} \\ & + \frac{2t^8(4r^4+7r^2t^2+4t^4)(4r^4+9r^2t^2+4t^4)}{(r^2+2t^2)^2(r^4+r^2t^2+t^4)^4}.\end{aligned} \tag{4.18}$$

As the Euler density (4.13), the signature density $\rho_\tau(M)$ in (4.18) is singular at the origin due to the first term but it is not a harmful singularity either because we will get a finite bulk contribution for the Hirzebruch signature $\tau(M)$. To verify it, consider the bulk part of the Hirzebruch signature $\tau(M)$ given by

$$\tau_{bulk}(M) = \frac{1}{2\pi^2}\int_M \rho_\tau(M) e^1 \wedge e^2 \wedge e^3 \wedge e^4 \tag{4.19}$$

and we get the same value $\tau_{bulk}(M) = -1$ for both instantons. The first boundary term in $\tau(M)$ vanishes for the same reason as in the Euler characteristic $\chi(M)$. And the eta-invariant $\eta_S(\partial M)$ is identically zero for the Eguchi-Hanson space because $\eta_S(\partial M)$ for k self-dual gravitational instantons is given by [53]

$$\eta_S(\partial M) = -\frac{2\epsilon}{3k} + \frac{(k-1)(k-2)}{3k} \tag{4.20}$$

where $\epsilon = 0$ for ALE boundary conditions and $\epsilon = 1$ for ALF boundary conditions. Although we did not explicitly calculate the eta-invariant $\eta_S(\partial M)$ for the BN instanton, it is

reasonable to expect that it will also vanish because the metric (3.43) for the BN instanton shows exactly the same asymptotic behavior as the Eguchi-Hanson space and it also satisfies the ALE boundary condition. Thus we conclude that

$$\tau(M) = -1 + 0 = -1 \tag{4.21}$$

for both NS and BN instantons.

Let us discuss some possible implications for the topological invariants of U(1) gauge fields. Our result (4.16) for the BN instanton strongly supports the topology change of spacetime speculated by Braden and Nekrasov [28]. Recall that the Euler characteristic $\chi(M)$ can be determined by the set of nuts and bolts through the fixed point theorem (see eq. (4.6) in [48])

$$\chi(M) = 2\,\sharp(\text{bolts}) + \sharp(\text{nuts}). \tag{4.22}$$

Therefore the result (4.16) implies that the BN instanton contains a non-contractible two-sphere $\mathbb{S}^2$ which is realized as a bolt in the gravitational solution (3.43) as we observed before. After such a blow up of $\mathbb{C}^2$ with $\mathbb{S}^2$, the resulting space becomes Kähler and we showed before that the metric (3.43) is indeed Kähler. Thus the emergent gravity approach provides a more accessible realization for the topology change of spacetime through U(1) instantons.

It might be emphasized again that symplectic U(1) gauge fields carry exactly the same topological invariants as four-manifolds. But those invariants are exotic from the gauge theory point of view because they are represented by higher derivative terms of U(1) field strength $F_{\mu\nu}$. (Actually this issue was posed before in the last of section 5 in [26].) Furthermore, in four dimensions, there exist two independent topological invariants, $\chi(M)$ and $\tau(M)$, while the second Chern class $c_2(E)$ is a unique topological invariant for the vector bundle E of gauge fields. Only for self-dual four-manifolds satisfying (2.37), two invariants are related to each other [50, 52]. For example, closed half-flat manifolds satisfy the relation $\chi(M) = \frac{3}{2}|\tau(M)|$ whereas noncompact half-flat manifolds obey $\chi(M) = |\tau(M)|+1$. But, for general four-manifolds, $\chi(M)$ and $\tau(M)$ are independent of each other. In addition, our explicit computation verifies that it is necessary to include not only bulk terms but also boundary contributions in order to get integer-valued topological invariants. All these features are unprecedented and quite bizarre from the gauge theory perspective and so further studies are required.

5. Discussion

Our circumstantial test of emergent gravity picture reveals that the NC spacetime (1.1) must be taken seriously as a pillar for quantum gravity. The symplectic structure of spacetime is the essence of emergent gravity realizing the duality between general relativity and NC U(1) gauge theory. We have to regard the NC algebra (1.1) as a raw precursor to the fabric of spacetime that is coalesced into an organized form that we recognize as spacetime. Our detailed analysis comparing the NS instanton and the BN instanton indicates that a spacetime singularity in general relativity can be resolved in the dual gauge theory description through the topology change of spacetime which is ample in NC spacetime.

Let us discuss why the topology change and the subsequent resolution of spacetime singularity are possible in NC spacetime. As we illustrated in this paper, the NC spacetime admits the emergent gravity in terms of NC U(1) gauge theory [2, 54]. In this description, spacetime geometry is defined by symplectic (or NC) U(1) gauge fields as exemplified by (2.14). Accordingly, the topology of spacetime is determined by the topology of U(1) gauge fields on NC spacetime. As is now well-known, the topology of U(1) gauge fields is nontrivial and rich [55, 56]. The pith of the nontrivial topology of NC U(1) gauge fields is the NC U(1) instantons satisfying the self-duality equation (2.21). And we observed that the nontrivial topology of U(1) instantons faithfully appears in the emergent gravity description. For example, NC U(1) instantons give rise to the ALE-type four-manifolds [29, 57] whose nontrivial topology is encoded in bolts (non-contractible two-cycles) while NC U(1) monopoles may be realized as the ALF-type four-manifolds whose nontrivial topology is encoded in nuts (isolated points). The nice formula (4.22) for the Euler characteristic clearly illuminates this aspect of four-manifolds emergent from symplectic (or NC) U(1) instantons or monopoles. Then a natural question is about the status of spacetime singularity in NC spacetime. It is worthwhile to notice that the NC space (1.1) is of the same kind as that of the Heisenberg algebra $[x^i, p_j] = i\hbar\delta^i_j$ in quantum mechanics where θ plays the role of $\hbar$. Thus one can expect that the NC effect will be significant in a deep microscopic scale, typically near the spacetime singularities. From the analogy between the NC spacetime and quantum mechanics, one can expect that there will be a vital spacetime uncertainty relation as an analogue of the famous Heisenberg's uncertainty relation $\Delta x \Delta p \geq \hbar$. This spacetime uncertainty relation gives rise to UV/IR mixing in NC gauge theory [58] and is responsible for the holographic principle in gravity [59, 60]. Therefore, in the NC space, it is impossible to localize a vast amount of energy to a point. Everything tends to spread out due to the spacetime exclusion. The best way to realize a localized object in NC spacetime is to construct a stable topological object such as NC instantons [25, 36] and GMS solitons [61]. But the topological objects in NC spacetime are regular solutions without any singularity and carry a nontrivial topology [55, 56]. As a result, if spacetime geometry is emergent from NC gauge fields, the spacetime singularity in general relativity may be a fake effect caused by our naive way of working in a purely commutative language.

Our result to expose the contrast between the NS instanton and the BN instanton seems to be useful to illuminate such aspects of spacetime singularity. So let us recapitulate the results for the NS and BN instantons, highlighting the role of NC spacetime. The NS instantons are obtained by the standard (undeformed) ADHM construction defined on the NC space (1.1). The NC effect automatically brings about the deformation of ADHM data and resolves the singularities of instanton moduli space [25]. The solution of the ADHM equations is completely regular and has the well-defined instanton number (4.5). The commutative description (3.15) of the NS instanton superficially looks singular but its gravitational description gives rise to a completely regular geometry (3.8) which is the simplest ALE gravitational instanton. And we have to think of the Eguchi-Hanson space (3.8) as already incorporating the backreaction of the NC instanton and we pointed out that this space contains a nontrivial two-cycle $\mathbb{S}^2$ known as the bolt. Therefore, after incorporating the backreaction of the NC instanton, the underlying space is changed from $\mathbb{R}^4$ to

an ALE space with the topology change. In the course of the transition, the Euler number $\chi(M)$ is accordingly changed from 1 (because of $\chi(\mathbb{R}^4) = 1$) to 2. On the other hand, the BN instanton is obtained by solving the deformed ADHM equations on commutative $\mathbb{C}^2$ (singling out a particular complex structure) [28]. Hence the deformation is not induced by a NC space but simply assumed to define the ADHM data. Nevertheless, the solution of the deformed ADHM equations is still singular. In order to make a non-singular solution, it is necessary to change the topology of spacetime. And we found that the gravitational solution determined by the BN instanton also contains a spacetime singularity. An interesting point is that, in spite of the spacetime singularity, the topological invariants of the BN instanton from the gravity point of view is exactly the same as the NS instanton. This reasoning thus brings a conclusion that the NC structure of spacetime must be taken seriously to resolve the spacetime singularity in general relativity.

We observed in (3.19) that NC gauge fields can be interpreted as the field variables defined in a locally inertial frame and their commutative description corresponds to the field variables in a laboratory frame represented in terms of general curvilinear coordinates. It presents a very beautiful picture about NC gauge fields. Moreover it turned out that this property holds even for the BN instanton which is neither self-dual nor anti-self-dual. Hence one may suspect that the formula (3.19) can be applied to generic NC gauge fields. Now we will prove the identity (3.19) for general U(1) gauge fields with the symmetric metric (2.14), i.e. $G_{\mu\nu} = G_{\nu\mu}$. Let us start with the standard formula

$$\frac{1}{4}\int d^4x\sqrt{G}G^{\mu\rho}G^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} = \frac{1}{2}\int F \wedge *F \tag{5.1}$$

where the Hodge $*$-operation is defined by

$$*F = \frac{1}{2}F_{\mu\nu} * (dx^\mu \wedge dx^\nu) = \frac{\sqrt{G}}{4}F_{\mu\nu}\varepsilon^{\mu\nu}{}_{\rho\sigma}dx^\rho \wedge dx^\sigma. \tag{5.2}$$

Using the fact that $F = \frac{1}{2}F_{ab}E^a \wedge E^b$ and $E^1 \wedge \cdots \wedge E^4 = \sqrt{G}d^4x$ where the one-form basis is defined by (2.30), one can write the right-hand side of (5.1) as

$$\frac{1}{2}\int F \wedge *F = \frac{1}{4}\int d^4x\sqrt{G}F_{ab}(x)F^{ab}(x). \tag{5.3}$$

Using the formula (2.6) for the Jacobian of the coordinate transformation (2.7) and comparing (2.16) with (5.1), we finally arrive at the result

$$\frac{1}{4}\int d^4y\widehat{F}_{ab}(y)\widehat{F}^{ab}(y) = \frac{1}{4}\int d^4yF_{ab}(y)F^{ab}(y). \tag{5.4}$$

The above result immediately implies the identity (3.19).

One may relax the condition that the metric (2.14) is symmetric and try to prove the identity (3.19) in a completely general context. So far we have been unable to prove the identity but we believe it is presumably true for arbitrary NC gauge fields. It will be interesting to confirm this conjecture.

Unfortunately the emergent gravity formulated by using the SW map as in section 2 cannot be applied to a general Riemannian metric for the following reason. First of all,

the effective metric defined by (2.14) is not symmetric in general. To have a symmetric Riemannian metric from symplectic gauge fields, the condition (2.26) has to be obeyed. This condition is reduced to the form (2.27) in a frame with $\theta^{\mu\nu} = \frac{\theta}{2}\eta^3_{\mu\nu}$ (that can always be achieved by performing an SO(4) rotation). Then some metric components in (2.14) identically vanish, e.g. $G_{12} = G_{34} = 0$. One can check that the Taub-NUT metric (3.34), for example, does not belong to such a class of metrics. Therefore we need a generalization to include a more general class of metrics in the bottom-up approach of emergent gravity. Recently we formulated in [22] such kind of general bottom-up approach for emergent gravity and its applications to general class of metrics will be addressed in our forthcoming paper [44].

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A. 't Hooft symbols

Since we heavily use several properties of the 't Hooft symbols, we reproduce here the appendix A in [30] for reader's convenience. The explicit components of the 't Hooft symbols $\eta^i_{\mu\nu}$ and $\overline{\eta}^i_{\mu\nu}$ for $i = 1, 2, 3$ are given by

$$\begin{aligned} \eta^i_{\mu\nu} &= \varepsilon^{i4\mu\nu} + \delta^{i\mu}\delta^{4\nu} - \delta^{i\nu}\delta^{4\mu}, \\ \overline{\eta}^i_{\mu\nu} &= \varepsilon^{i4\mu\nu} - \delta^{i\mu}\delta^{4\nu} + \delta^{i\nu}\delta^{4\mu} \end{aligned} \tag{A.1}$$

with $\varepsilon^{1234} = 1$. They satisfy the following relations

$$\eta^{(\pm)i}_{\mu\nu} = \pm\frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}\eta^{(\pm)i}_{\rho\sigma}, \tag{A.2}$$

$$\eta^{(\pm)i}_{\mu\nu}\eta^{(\pm)i}_{\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} \pm \varepsilon_{\mu\nu\rho\sigma}, \tag{A.3}$$

$$\varepsilon_{\mu\nu\rho\sigma}\eta^{(\pm)i}_{\sigma\lambda} = \mp(\delta_{\lambda\rho}\eta^{(\pm)i}_{\mu\nu} + \delta_{\lambda\mu}\eta^{(\pm)i}_{\nu\rho} - \delta_{\lambda\nu}\eta^{(\pm)i}_{\mu\rho}), \tag{A.4}$$

$$\eta^{(\pm)i}_{\mu\nu}\eta^{(\mp)j}_{\mu\nu} = 0, \tag{A.5}$$

$$\eta^{(\pm)i}_{\mu\rho}\eta^{(\pm)j}_{\nu\rho} = \delta^{ij}\delta_{\mu\nu} + \varepsilon^{ijk}\eta^{(\pm)k}_{\mu\nu}, \tag{A.6}$$

$$\eta^{(\pm)i}_{\mu\rho}\eta^{(\mp)j}_{\nu\rho} = \eta^{(\pm)i}_{\nu\rho}\eta^{(\mp)j}_{\mu\rho}, \tag{A.7}$$

$$\varepsilon^{ijk}\eta^{(\pm)j}_{\mu\nu}\eta^{(\pm)k}_{\rho\sigma} = \delta_{\mu\rho}\eta^{(\pm)i}_{\nu\sigma} - \delta_{\mu\sigma}\eta^{(\pm)i}_{\nu\rho} - \delta_{\nu\rho}\eta^{(\pm)i}_{\mu\sigma} + \delta_{\nu\sigma}\eta^{(\pm)i}_{\mu\rho}, \tag{A.8}$$

where $\eta^{(+)i}_{\mu\nu} \equiv \eta^i_{\mu\nu}$ and $\eta^{(-)i}_{\mu\nu} \equiv \overline{\eta}^i_{\mu\nu}$.

If we introduce two families of 4×4 matrices defined by

$$[T^i_+]_{\mu\nu} \equiv \eta^i_{\mu\nu}, \qquad [T^i_-]_{\mu\nu} \equiv \overline{\eta}^i_{\mu\nu}, \tag{A.9}$$

the matrices in (A.9) provide two independent spin $s = \frac{3}{2}$ representations of SU(2) Lie algebra. Explicitly, they are given by

$$T_+^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad T_+^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad T_+^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{A.10}$$

$$T_-^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad T_-^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad T_-^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{A.11}$$

according to the definition (A.1). The matrices in (A.6) and (A.7) immediately show that $T_\pm^i$ satisfy SU(2) Lie algebras, i.e.,

$$[T_\pm^i, T_\pm^j] = -2\varepsilon^{ijk} T_\pm^k, \qquad [T_\pm^i, T_\mp^j] = 0. \tag{A.12}$$

B. Spin connections and curvature tensors

In this appendix, we calculate the spin connections and curvature tensors for the metric (3.1). The spin connections are determined by solving the torsion free condition

$$T^a \equiv de^a + \omega^a{}_b \wedge e^b = 0. \tag{B.1}$$

Explicitly they are given by

$$\begin{aligned} \omega_{14} &= \frac{B'r + B}{A}\sigma^1, & \omega_{12} &= \frac{A^2 - 2B^2}{B^2}\sigma^3, \\ \omega_{24} &= \frac{B'r + B}{A}\sigma^2, & \omega_{31} &= -\frac{A}{B}\sigma^2, \\ \omega_{34} &= \frac{A'r + A}{A}\sigma^3, & \omega_{23} &= -\frac{A}{B}\sigma^1, \end{aligned} \tag{B.2}$$

where $' \equiv \frac{d}{dr}$. The curvature tensor is then defined by

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{B.3}$$

The explicit results are given by

$$\begin{aligned} R_{12} &= \frac{1}{r^2B^2}\Big[4 - 3\Big(\frac{A}{B}\Big)^2 - \Big(\frac{B'r + B}{A}\Big)^2\Big] e^1 \wedge e^2 + \frac{2}{rAB^3}(AB' - A'B) e^3 \wedge e^4, \\ R_{31} &= \frac{1}{r^2B^2}\Big[\Big(\frac{A}{B}\Big)^2 - \frac{(A'r + A)(B'r + B)B}{A^3}\Big] e^3 \wedge e^1 - \frac{1}{rAB^3}(AB' - A'B) e^2 \wedge e^4, \\ R_{14} &= \frac{rA'B' + A'B - 2AB' - rAB''}{rA^3B} e^1 \wedge e^4 + \frac{1}{r^2B^2}\Big[\frac{A'r + A}{A} - \frac{B'r + B}{B}\Big] e^2 \wedge e^3, \\ R_{23} &= \frac{1}{r^2B^2}\Big[\Big(\frac{A}{B}\Big)^2 - \frac{(A'r + A)(B'r + B)B}{A^3}\Big] e^2 \wedge e^3 - \frac{1}{rAB^3}(AB' - A'B) e^1 \wedge e^4, \\ R_{24} &= \frac{rA'B' + A'B - 2AB' - rAB''}{rA^3B} e^2 \wedge e^4 + \frac{1}{r^2B^2}\Big[\frac{A'r + A}{A} - \frac{B'r + B}{B}\Big] e^3 \wedge e^1, \\ R_{34} &= -\frac{2}{r^2B^2}\Big[\frac{A'r + A}{A} - \frac{B'r + B}{B}\Big] e^1 \wedge e^2 + \frac{1}{rA^4}\big(r(A')^2 - rAA'' - AA'\big) e^3 \wedge e^4, \end{aligned} \tag{B.4}$$

where $'' \equiv \frac{d^2}{dr^2}$.

Using the above results, one can calculate the following quantities:

$$\begin{aligned}
\rho_\chi(M) &\equiv \frac{1}{64}\varepsilon^{abcd}\varepsilon^{efgh}R_{abef}R_{cdgh} \\
&= \frac{1}{2}\big(R_{1234}^2 + R_{1423}^2 + R_{2431}^2 + R_{1212}R_{3434} + R_{3131}R_{2424} + R_{1414}R_{2323}\big) \\
&= \frac{1}{2r^4A^6B^6}\Big[rB^2\Big(3A^4 - 4A^2B^2 + B^2\big(B + rB'\big)^2\Big)\Big(-r(A')^2 + A\big(A' + rA''\big)\Big) \\
&\quad +2rB\Big(A^5 - B^3\big(A + rA'\big)\big(B + rB'\big)\Big)\Big(-2AB' + A'(B + rB') - rAB''\Big) \\
&\quad +6r^2A^4\big(AB' - A'B\big)^2\Big],
\end{aligned} \tag{B.5}$$

and

$$\begin{aligned}
\rho_\tau(M) &\equiv \frac{1}{48}\varepsilon^{cdef}R_{abcd}R_{abef} \\
&= \frac{1}{3}\big(R_{1212}R_{1234} + R_{1313}R_{2431} + R_{1414}R_{1423} + R_{2323}R_{2314} + R_{2424}R_{2431} + R_{3434}R_{1234}\big) \\
&= \frac{2(AB' - A'B)}{3r^3A^5B^7}\Big[rA^2B^2\Big(-r(B')^2 + B\big(B' + rB''\big)\Big) - 4A^4\big(A^2 - B^2\big) \\
&\quad +rB^4\Big(r(A')^2 - A\big(A' + rA''\big)\Big)\Big].
\end{aligned} \tag{B.6}$$

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